Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Special Instructions/Useful Data	
	All angles are in radian
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n): x_i \in \mathbb{R}, 1 \le i \le n\}$
M^T	Transpose of the matrix M
f'	Derivative of the function f
$P(E)$	Probability of the event E
$E(X)$	Expectation of the random variable X
$Var(X)$	Variance of the random variable X
i.i.d.	Independently and identically distributed
$U(a,b)$	Continuous uniform distribution on $(a,b), -\infty < a < b < \infty$
$\Phi(a)$	$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx$
$\Gamma(p)$	The gamma function $\Gamma(p) = \int_0^\infty e^{-t}\, t^{p-1}\,dt, \quad p > 0$
$n!$	The factorial function $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 2$ and, for $n \geq 1$,

$$a_{n+1} = \frac{2\,a_n + 1}{a_n + 1}.$$

Then

(A) $1.5 \leq a_n \leq 2$, for all natural number $n \geq 1$

(B) there exists a natural number $n \geq 1$ such that $a_n > 2$

(C) there exists a natural number $n \geq 1$ such that $a_n < 1.5$

(D) there exists a natural number $n \geq 1$ such that $a_n = \frac{1+\sqrt{5}}{2}$

Q.2 The value of

$$\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{n^2} e^{-2n}$$

is

(A) e^{-2} (B) e^{-1} (C) e (D) e^2

Q.3 Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$be two convergent sequences of real numbers. For $n \geq 1$, define $u_n = \max\{a_n, b_n\}$ and $v_n = \min\{a_n, b_n\}$. Then

(A) neither $\{u_n\}_{n\geq 1}$ nor $\{v_n\}_{n\geq 1}$ converges

(B) $\{u_n\}_{n\geq 1}$ converges but $\{v_n\}_{n\geq 1}$ does not converge

(C) $\{u_n\}_{n\geq 1}$ does not converge but $\{v_n\}_{n\geq 1}$ converges

(D) both $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ converge

Q.4 Let $M = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$. If I is the 2×2 identity matrix and $\mathbf{0}$ is the 2×2 zero matrix, then

(A) $20\,M^2 - 13\,M + 7\,I = \mathbf{0}$

(B) $20\,M^2 - 13\,M - 7\,I = \mathbf{0}$

(C) $20\,M^2 + 13\,M + 7\,I = \mathbf{0}$

(D) $20\,M^2 + 13\,M - 7\,I = \mathbf{0}$

Q.5 Let X be a random variable with the probability density function

$$f(x) = \begin{cases} \frac{\alpha^p}{\Gamma(p)} e^{-\alpha x} x^{p-1}, & x \geq 0, \alpha > 0, p > 0, \\ 0, & \text{otherwise.} \end{cases}$$

If $E(X) = 20$ and $Var(X) = 10$, then (α, p) is

(A) $(2, 20)$ (B) $(2, 40)$ (C) $(4, 20)$ (D) $(4, 40)$

Q.6 Let X be a random variable with the distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4} + \frac{4x - x^2}{8}, & 0 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Then

$$P(X = 0) + P(X = 1.5) + P(X = 2) + P(X \geq 1)$$

equals

(A) $\frac{3}{8}$ (B) $\frac{5}{8}$ (C) $\frac{7}{8}$ (D) 1

Q.7 Let X_1, X_2 and X_3 be i.i.d. $U(0, 1)$ random variables. Then $E\left(\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right)$ equals

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{4}$

Q.8 Let $x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$ and $x_5 = 0$ be the observed values of a random sample of size 5 from a discrete distribution with the probability mass function

$$f(x; \theta) = P(X = x) = \begin{cases} \frac{\theta}{3}, & x = 0, \\ \frac{2\theta}{3}, & x = 1, \\ \frac{1 - \theta}{2}, & x = 2,3, \end{cases}$$

where $\theta \in [0, 1]$ is the unknown parameter. Then the maximum likelihood estimate of θ is

(A) $\frac{2}{5}$ (B) $\frac{3}{5}$ (C) $\frac{5}{7}$ (D) $\frac{5}{9}$

Q.9 Consider four coins labelled as $1, 2, 3$ and 4. Suppose that the probability of obtaining a 'head' in a single toss of the i^{th} coin is $\frac{i}{4}, i = 1, 2, 3, 4$. A coin is chosen uniformly at random and flipped. Given that the flip resulted in a 'head', the conditional probability that the coin was labelled either 1 or 2 equals

(A) $\frac{1}{10}$ (B) $\frac{2}{10}$ (C) $\frac{3}{10}$ (D) $\frac{4}{10}$

Q.10 Consider the linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i; i = 1, 2, \ldots, n$, where ϵ_i's are i.i.d. standard normal random variables. Given that

$$\frac{1}{n}\sum_{i=1}^{n} x_i = 3.2, \quad \frac{1}{n}\sum_{i=1}^{n} y_i = 4.2, \quad \frac{1}{n}\sum_{j=1}^{n}\left(x_j - \frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 = 1.5 \text{ and}$$

$$\frac{1}{n}\sum_{j=1}^{n}\left(x_j - \frac{1}{n}\sum_{i=1}^{n} x_i\right)\left(y_j - \frac{1}{n}\sum_{i=1}^{n} y_i\right) = 1.7,$$

the maximum likelihood estimates of β_0 and β_1, respectively, are

(A) $\frac{17}{15}$ and $\frac{32}{75}$ (B) $\frac{32}{75}$ and $\frac{17}{15}$

(C) $\frac{17}{15}$ and $\frac{43}{75}$ (D) $\frac{43}{75}$ and $\frac{17}{15}$

Q. 11 – Q. 30 carry two marks each.

Q.11 Let $f: [-1,\ 1] \to \mathbb{R}$ be defined by $f(x) = \frac{x^2 + [\sin \pi x]}{1+|x|}$, where $[y]$ denotes the greatest integer less than or equal to y. Then

(A) f is continuous at $-\frac{1}{2}$,0, 1

(B) f is discontinuous at $-1, 0, \frac{1}{2}$

(C) f is discontinuous at $-1, -\frac{1}{2}, 0, \frac{1}{2}$

(D) f is continuous everywhere except at 0

Q.12 Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - \frac{\cos x}{2}$ and $g(x) = \frac{x \sin x}{2}$. Then

(A) $f(x) = g(x)$ for more than two values of x

(B) $f(x) \neq g(x)$, for all x in $\mathbb{R}$

(C) $f(x) = g(x)$ for exactly one value of x

(D) $f(x) = g(x)$ for exactly two values of x

Q.13 Consider the domain $D = \{ (x, y) \in \mathbb{R}^2 : x \leq y \}$ and the function $h: D \to \mathbb{R}$ defined by

$$h\big((x, y)\big) = (x-2)^4 + (y-1)^4, (x, y) \in D.$$

Then the minimum value of h on D equals

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{1}{16}$

Q.14 Let $M = [X \quad Y \quad Z]$ be an orthogonal matrix with $X, Y, Z \in \mathbb{R}^3$ as its column vectors. Then

$$Q = X\,X^T + Y\,Y^T$$

(A) is a skew-symmetric matrix

(B) is the 3×3 identity matrix

(C) satisfies $Q^2 = Q$

(D) satisfies $QZ = Z$

Q.15 Let $f: [0, 3] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ e^{x^2} - e, & 1 \leq x < 2 \\ e^{x^2} + 1, & 2 \leq x \leq 3. \end{cases}$$

Now, define $F: [0, 3] \rightarrow \mathbb{R}$ by

$$F(0) = 0 \text{ and } F(x) = \int_0^x f(t)dt, \text{for } 0 < x \leq 3.$$

Then

(A) F is differentiable at $x = 1$ and $F'(1) = 0$

(B) F is differentiable at $x = 2$ and $F'(2) = 0$

(C) F is not differentiable at $x = 1$

(D) F is differentiable at $x = 2$ and $F'(2) = 1$

Q.16 If x, y and z are real numbers such that $4\,x + 2\,y + z = 31$ and $2\,x + 4\,y - z = 19$, then the value of $9\,x + 7\,y + z$

(A) cannot be computed from the given information

(B) equals $\frac{281}{3}$

(C) equals $\frac{182}{3}$

(D) equals $\frac{218}{3}$

Q.17 Let $M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}$. If

$$V = \left\{(x, y, 0) \in \mathbb{R}^3 : M \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \text{ and } W = \left\{(x, y, z) \in \mathbb{R}^3 : M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\},$$

then

(A) the dimension of V equals 2

(B) the dimension of W equals 2

(C) the dimension of V equals 1

(D) $V \cap W = \{(0,0,0)\}$

Q.18 Let M be a 3×3 non-zero, skew-symmetric real matrix. If I is the 3×3 identity matrix, then

(A) M is invertible

(B) the matrix $I + M$ is invertible

(C) there exists a non-zero real number α such that $\alpha I + M$ is not invertible

(D) all the eigenvalues of M are real

Q.19 Let X be a random variable with the moment generating function

$$M_X(t) = \frac{6}{\pi^2} \sum_{n \geq 1} \frac{e^{t^2/2n}}{n^2},\ t \in \mathbb{R}.$$

Then $P(X \in \mathbb{Q})$, where $\mathbb{Q}$ is the set of rational numbers, equals

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

Q.20 Let X be a discrete random variable with the moment generating function

$$M_X(t) = \frac{(1 + 3\,e^t)^2 (3 + e^t)^3}{1024}, t \in \mathbb{R}.$$

Then

(A) $E(X) = \frac{9}{4}$ (B) $Var(X) = \frac{15}{32}$

(C) $P(X \geq 1) = \frac{27}{1024}$ (D) $P(X = 5) = \frac{3}{1024}$

Q.21 Let $\{X_n\}_{n \geq 1}$ be a sequence of independent random variables with X_n having the probability density function as

$$f_n(x) = \begin{cases} \dfrac{1}{2^{n/2}\ \Gamma(\frac{n}{2})}\ e^{-\frac{x}{2}}\ \ x^{(\frac{n}{2}-1)}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{n \to \infty} \left[P\left(X_n > \frac{3}{4} n\right) + P(\,X_n > n + 2\sqrt{2n}) \right]$$

equals

(A) $1 + \Phi(2)$ (B) $1 - \Phi(2)$ (C) $\Phi(2)$ (D) $2 - \Phi(2)$

Q.22 Let X be a Poisson random variable with mean $\frac{1}{2}$. Then $E((X+1)!)$ equals

(A) $2\,e^{-\frac{1}{2}}$ (B) $4\,e^{-\frac{1}{2}}$ (C) $4\,e^{-1}$ (D) $2\,e^{-1}$

Q.23 Let X be a standard normal random variable. Then $P(X^3 - 2\,X^2 - X + 2 > 0)$ equals

(A) $2\Phi(1) - 1$ (B) $1 - \Phi(2)$

(C) $2\Phi(1) - \Phi(2)$ (D) $\Phi(2) - \Phi(1)$

Q.24 Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} 2, & 0 \le x \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $a = E(Y|X = \frac{1}{2})$ and $b = Var(Y|X = \frac{1}{2})$. Then (a, b) is

(A) $\left(\frac{3}{4}, \frac{7}{12}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{48}\right)$

(C) $\left(\frac{1}{4}, \frac{7}{12}\right)$ (D) $\left(\frac{3}{4}, \frac{1}{48}\right)$

Q.25 Let X and Y have the joint probability mass function

$$P(X = m, Y = n) = \begin{cases} \dfrac{m+n}{21}, & m = 1,2,3; n = 1,2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(X = 2|Y = 2)$ equals

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Q.26 Let X and Y be two independent standard normal random variables. Then the probability density function of $Z = \frac{|X|}{|Y|}$ is

(A) $f(z) = \begin{cases} \frac{\sqrt{1/2}}{\sqrt{\pi}}\, e^{-\frac{z}{2}}\, z^{-\frac{1}{2}}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$

(B) $f(z) = \begin{cases} \frac{2}{\sqrt{2\pi}}\, e^{-z^2/2}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$

(C) $f(z) = \begin{cases} e^{-z}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$

(D) $f(z) = \begin{cases} \frac{2}{\pi}\, \frac{1}{(1+z^2)}, & z > 0, \\ 0, & \text{otherwise} \end{cases}$

Q.27 Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} e^{-y}, & 0 < x < y < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Then the correlation coefficient between X and Y equals

(A) $\frac{1}{3}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{\sqrt{3}}$

Q.28 Let $x_1 = -2, x_2 = 1$ and $x_3 = -1$ be the observed values of a random sample of size three from a discrete distribution with the probability mass function

$$f(x;\theta) = P(X = x) = \begin{cases} \dfrac{1}{2\,\theta + 1}, & x \in \{-\theta, -\theta + 1, \dots, 0, \dots, \theta\}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in \Theta = \{1, 2, \dots\}$ is the unknown parameter. Then the method of moment estimate of θ is

(A) 1 (B) 2 (C) 3 (D) 4

Q.29 Let X be a random sample from a discrete distribution with the probability mass function

$$f(x;\theta) = P(X = x) = \begin{cases} \dfrac{1}{\theta}, & x = 1, 2, \dots, \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in \Theta = \{20, 40\}$ is the unknown parameter. Consider testing

$$H_0: \theta = 40 \text{ against } H_1: \theta = 20$$

at a level of significance $\alpha = 0.1$. Then the uniformly most powerful test rejects H_0 if and only if

(A) $X \le 4$ (B) $X > 4$

(C) $X \ge 3$ (D) $X < 3$

Q.30 Let X_1 and X_2 be a random sample of size 2 from a discrete distribution with the probability mass function

$$f(x;\theta) = P(X = x) = \begin{cases} \theta, & x = 0, \\ 1 - \theta, & x = 1, \end{cases}$$

where $\theta \in \Theta = \{0.2, 0.4\}$ is the unknown parameter. For testing $H_0: \theta = 0.2$ against $H_1: \theta = 0.4$, consider a test with the critical region

$$C = \{(x_1, x_2) \in \{0,1\} \times \{0,1\} : x_1 + x_2 < 2\}.$$

Let α and β denote the probability of Type I error and power of the test, respectively.
Then (α, β) is

(A) $(0.36, 0.74)$ (B) $(0.64, 0.36)$

(C) $(0.05, 0.64)$ (D) $(0.36, 0.64)$

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that

$$a_n = \sum_{k=n+1}^{2n} \frac{1}{k}, \quad n \geq 1.$$

Then which of the following statement(s) is (are) true?

(A) $\{a_n\}_{n\geq 1}$ is an increasing sequence

(B) $\{a_n\}_{n\geq 1}$ is bounded below

(C) $\{a_n\}_{n\geq 1}$ is bounded above

(D) $\{a_n\}_{n\geq 1}$ is a convergent sequence

Q.32 Let $\sum_{n\geq 1} a_n$ be a convergent series of positive real numbers. Then which of the following statement(s) is (are) true?

(A) $\sum_{n\geq 1}(a_n)^2$ is always convergent

(B) $\sum_{n\geq 1}\sqrt{a_n}$ is always convergent

(C) $\sum_{n\geq 1}\frac{\sqrt{a_n}}{n}$ is always convergent

(D) $\sum_{n\geq 1}\frac{\sqrt{a_n}}{n^{1/4}}$ is always convergent

Q.33 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 3$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n^2 - 2\,a_n + 4}{2}.$$

Then which of the following statement(s) is (are) true?

(A) $\{a_n\}_{n\geq 1}$ is a monotone sequence

(B) $\{a_n\}_{n\geq 1}$ is a bounded sequence

(C) $\{a_n\}_{n\geq 1}$ does not have finite limit, as $n \to \infty$

(D) $\lim_{n\to\infty} a_n = 2$

Q.34 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^4\left(2 + \sin\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then which of the following statement(s) is (are) true?

(A) f attains its minimum at 0

(B) f is monotone

(C) f is differentiable at 0

(D) $f(x) > 2\,x^4 + x^3$, for all $x > 0$

Q.35 Let P be a probability function that assigns the same weight to each of the points of the sample space $\Omega = \{1,2,3,4\}$. Consider the events $E = \{1,2\}, F = \{1,3\}$ and $G = \{3,4\}$. Then which of the following statement(s) is (are) true?

(A) E and F are independent

(B) E and G are independent

(C) F and G are independent

(D) E, F and G are independent

Q.36 Let $X_1, X_2, \dots, X_n, n \geq 5$, be a random sample from a distribution with the probability density function

$$f(x;\theta) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in \mathbb{R}$ is the unknown parameter. Then which of the following statement(s) is (are) true?

(A) A 95% confidence interval of θ has to be of finite length

(B) $\left(\min\{X_1, X_2, \dots, X_n\} + \frac{1}{n}\ln(0.05),\ \min\{X_1, X_2, \dots, X_n\}\right)$ is a 95% confidence interval of θ

(C) A 95% confidence interval of θ can be of length 1

(D) A 95% confidence interval of θ can be of length 2

Q.37 Let $X_1, X_2, \dots, X_n$ be a random sample from $U(0,\theta)$, where $\theta > 0$ is the unknown parameter. Let $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$. Then which of the following is (are) consistent estimator(s) of θ^3?

(A) $8\,X_n^3$

(B) $X_{(n)}^3$

(C) $\left(\frac{2}{n}\sum_{i=5}^{n} X_i\right)^3$

(D) $\frac{n\,X_{(n)}^3 + 1}{n+1}$

Q.38 Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with the probability density function

$$f(x;\theta) = \begin{cases} c(\theta)\, e^{-(x-\theta)}, & x \geq 2\theta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta \in \mathbb{R}$ is the unknown parameter. Then which of the following statement(s) is (are) true?

(A) The maximum likelihood estimator of θ is $\frac{\min\{X_1, X_2, \dots, X_n\}}{2}$

(B) $c(\theta) = 1$, for all $\theta \in \mathbb{R}$

(C) The maximum likelihood estimator of θ is $\min\{X_1, X_2, \dots, X_n\}$

(D) The maximum likelihood estimator of θ does not exist

Q.39 Let $X_1, X_2, \dots, X_n$ be a random sample from a distribution with the probability density function

$$f(x;\theta) = \begin{cases} \theta^2 x\, e^{-\theta x}, & x > 0, \\ 0, & \text{otherwise,} \end{cases}$$

where $\theta > 0$ is the unknown parameter. If $Y = \sum_{i=1}^{n} X_i$, then which of the following statement(s) is (are) true?

(A) Y is a complete sufficient statistic for θ

(B) $\frac{2n}{Y}$ is the uniformly minimum variance unbiased estimator of θ

(C) $\frac{2n-1}{Y}$ is the uniformly minimum variance unbiased estimator of θ

(D) $\frac{2n+1}{Y}$ is the uniformly minimum variance unbiased estimator of θ

Q.40 Let $X_1, X_2, \dots, X_n$ be a random sample from $U(\theta, \theta+1)$, where $\theta \in \mathbb{R}$ is the unknown parameter. Let $U = \max\{X_1, X_2, \dots, X_n\}$ and $V = \min\{X_1, X_2, \dots, X_n\}$. Then which of the following statement(s) is (are) true?

(A) U is a consistent estimator of θ

(B) V is a consistent estimator of θ

(C) $2U - V - 2$ is a consistent estimator of θ

(D) $2V - U + 1$ is a consistent estimator of θ

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that

$$a_n = \frac{1+3+5+\cdots+(2n-1)}{n!}, \quad n \geq 1.$$

Then $\sum_{n\geq 1} a_n$ converges to ____________

Q.42 Let

$$S = \left\{(x, y) \in \mathbb{R}^2 : x, y \geq 0, \qquad \sqrt{4-(x-2)^2} \leq y \leq \sqrt{9-(x-3)^2}\right\}.$$

Then the area of S equals ____________

Q.43 Let $S = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$. Then the area of S equals ____________

Q.44 Let

$$J = \frac{1}{\pi}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}\, dt\,.$$

Then the value of J equals ____________

Q.45 A fair die is rolled three times independently. Given that 6 appeared at least once, the conditional probability that 6 appeared exactly twice equals ____________

Q.46 Let X and Y be two positive integer valued random variables with the joint probability mass function

$$P(X=m, Y=n) = \begin{cases} g(m)\, h(n), & m, n \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $g(m) = \left(\frac{1}{2}\right)^{m-1}, m \geq 1$ and $h(n) = \left(\frac{1}{3}\right)^{n}, n \geq 1$. Then $E(X\,Y)$ equals ____________

Q.47 Let E, F and G be three events such that

$$P(E \cap F \cap G) = 0.1, P(G|F) = 0.3 \text{ and } P(E|F \cap G) = P(E|F).$$

Then $P(G|E \cap F)$ equals ____________

Q.48 Let A_1, A_2 and A_3 be three events such that

$$P(A_i) = \frac{1}{3}, i = 1,2,3;\ P\left(A_i \cap A_j\right) = \frac{1}{6}, 1 \leq i \neq j \leq 3 \text{ and } P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}.$$

Then the probability that none of the events A_1, A_2, A_3 occur equals ____________

Q.49 Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with the probability density function

$$f(x) = \frac{1}{4}\ e^{-|x-4|} + \frac{1}{4} e^{-|x-6|}, x \in \mathbb{R}.$$

Then $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to ____________

Q.50 Let $x_1 = 1.1, x_2 = 2.2$ and $x_3 = 3.3$ be the observed values of a random sample of size three from a distribution with the probability density function

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $\theta \in \Theta = \{1, 2, \ldots\}$ is the unknown parameter. Then the maximum likelihood estimate of θ equals ____________

Q. 51 – Q. 60 carry two marks each.

Q.51 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous on $\mathbb{R}$ with $f'(3) = 18$. Define

$$g_n(x) = n\left(f\left(x+\frac{5}{n}\right) - f\left(x-\frac{2}{n}\right)\right).$$

Then $\lim_{n\to\infty} g_n(3)$ equals ___________

Q.52 Let $M = \sum_{i=1}^{4} X_i X_i^T$, where

$X_1^T = [1 \quad -1 \quad 1 \quad 0]$, $X_2^T = [1 \quad 1 \quad 0 \quad 1]$, $X_3^T = [1 \quad 3 \quad 1 \quad 0]$ and $X_4^T = [1 \quad 1 \quad 1 \quad 0]$.

Then the rank of M equals ____________

Q.53 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} f'(x) = 2$. Then

$$\lim_{x\to\infty}\left(1+\frac{f(x)}{x^2}\right)^x$$

equals ____________

Q.54 The value of

$$\int_0^{\frac{\pi}{2}}\left(\int_0^x e^{\sin y}\sin x \; dy\right) dx$$

equals ____________

Q.55 Let X be a random variable with the probability density function

$$f(x) = \begin{cases} 4\,x^k, & 0 < x < 1, \\ x - \dfrac{x^2}{2}, & 1 \le x < 2, \\ 0, & \text{otherwise,} \end{cases}$$

where k is a positive integer. Then $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ equals ____________

Q.56 Let X and Y be two discrete random variables with the joint moment generating function

$$M_{X,Y}(t_1, t_2) = \left(\frac{1}{3}\, e^{t_1} + \frac{2}{3}\right)^2 \left(\frac{2}{3}\, e^{t_2} + \frac{1}{3}\right)^3, t_1, t_2 \in \mathbb{R}.$$

Then $P(2X + 3Y > 1)$ equals ____________

Q.57 Let X_1, X_2, X_3 and X_4 be i.i.d. discrete random variables with the probability mass function

$$P(X_1 = n) = \begin{cases} \frac{3^{n-1}}{4^n}, & n = 1,2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(X_1 + X_2 + X_3 + X_4 = 6)$ equals ____________

Q.58 Let X be a random variable with the probability mass function

$$P(X = n) = \begin{cases} \frac{1}{10}, & n = 1,2, \cdots, 10, \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(\max\{X, 5\})$ equals ____________

Q.59 Let X be a sample observation from $U(\theta, \theta^2)$ distribution, where $\theta \in \Theta = \{2,3\}$ is the unknown parameter. For testing

$$H_0: \theta = 2 \text{ against } H_1: \theta = 3,$$

let α and β be the size and power, respectively, of the test that rejects H_0 if and only if $X \geq 3.5$. Then $\alpha + \beta$ equals ____________

Q.60 A fair die is rolled four times independently. For $i = 1, 2, 3, 4$, define

$$Y_i = \begin{cases} 1, & \text{if 6 appears in the } i^{th} \text{ throw,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $P(\max\{Y_1, Y_2, Y_3, Y_4\} = 1)$ equals ____________

END OF THE QUESTION PAPER