Non-CS Group (Mathematics)

Note: Answer all questions from Part-A and any eight questions from Part-B.

Part-A

NC1. Suppose a crime has been committed and there are three suspects: Professor Plum, Mrs. Peacock, and Mr. Green. Given that:

- At least one of the above three suspects is guilty.
- Not all of them are guilty.
- If Mrs. Peacock is guilty, then so is Professor Plum.
- If Mr. Green is innocent, then so is Professor Plum.

Prove or disprove each of the following statements:

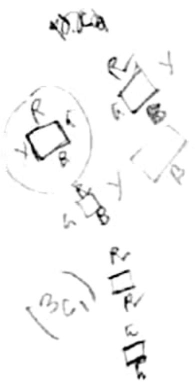
(i) Mr. Green is guilty.

(ii) Mrs. Peacock is innocent.

[3+3 = 6]

NC2. A room has four walls. In how many different ways can you color the four walls, using colors from the set {Red, Green, Blue, Yellow}, such that no two adjacent walls have the same color?

[7]



NC3. Consider a round-robin football tournament among $n \geq 3$ teams, where every team plays a match with every other team exactly once. For any team, each match results in a *win*, *draw* or *loss*. In the tournament, each team loses at least one match. Prove or disprove the following statement:

There must be two teams with the same number of wins at the end of the tournament.

[7]

Part-B

NC4. Consider an $n \times n$ grid as follows:

$$S = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} \quad 1 \leq i, j \leq n\}.$$

A triangle with its vertices on the grid points of S is *nice* if

- the area of the triangle is 3, and
- at least one of its edges is parallel to either x-axis or y-axis.

Find the number of nice triangles. [10]

NC5. (i) Let $f : [0, 1] \to [0, 1]$ be a strictly decreasing continuous function with $f(0) = 1$ and $f(1) = 0$. Show that

$$\int_0^1 f(x)dx = \int_0^1 f^{-1}(y)dy.$$

f(x)+C=

(ii) Let f be a differentiable function on $[-2, 2]$ such that $f(-2) = 1$, $f(2) = 5$ and $|f'(x)| < 1$ for all $x \in [-2, 2]$

Part-B

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[6+4 = 10]

$f(x)+C=$

$f(2)$

$\boxed{f(0)=2}$

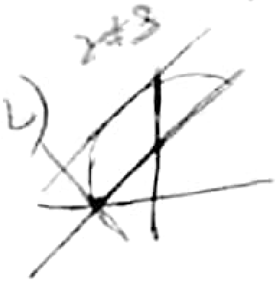
$f'(x)=1$

$f(x)=x+C$

$2+C=3$

$x+3$

$(x+2)$



NC6. (i) Consider a simple, undirected, connected graph $G = (V, E)$ whose every vertex has degree at least 2. Show that G contains a cycle.

(ii) A simple, undirected, connected graph is 2-*edge connected* if at least 2 edges need to be removed to make G disconnected.

(a) Show an example of a tree with 11 vertices that can be converted to a 2-*edge connected* graph by adding an edge.

(b) Show an example of a tree with 11 vertices that can be converted to a 2-*edge connected* graph by adding five edges.

[5+(2+3) = 10]

NC7. Let $a_0, a_1, \ldots, a_n$ be real numbers with property that

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Prove that the equation

$$a_0 + a_1 x + \cdots + a_n x^n = 0$$

has a solution in $(0, 1)$. [10]

NC8. Find the largest positive real number δ such that, for real numbers x and y

$$|x - y| < \delta \text{ implies } |\cos x - \cos y| < \sqrt{2}.$$

[10]

NC9. Let $f(x)$ be a nonconstant polynomial with real coefficients. If there exists a complex root α of $f(x)$ with multiplicity greater than 1, show that the polynomial $f(f(x))$ also has a complex root with multiplicity greater than 1. [10]

NC10. A particular Covid-19 vaccine produces a side effect (fever) with probability $p \in [0, 1]$. A clinic vaccinates 500 people each day and records the number of people having side effects on 5 randomly chosen days.

(i) What is the probability that the number of people who developed side effects on the 5 observed days are 10, 15, 0, 20, 25?

(ii) Find the value of p that maximizes the probability obtained in part (i).

(iii) Based on the value of p obtained in part (ii), what is the expected number of people who will develop side effects on each day?

[4+4+2 = 10]

NC11. Let $A, A_1, A_2, \ldots, A_m$ be $n \times n$ matrices with real entries s[...]
that

$$A = \sum_{i=1}^{m} A_i.$$

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[4+4+2 = 10]

NC11. Let $A, A_1, A_2, \ldots, A_m$ be $n \times n$ matrices with real entries such that

$$A = \sum_{t=1}^{m} A_t.$$

(i) Show that

$$\text{rank}(A) \leq \sum_{t=1}^{m} \text{rank}(A_t).$$

(ii) An $n \times n$ matix B with real entries is defined as a *two-block matrix* if there exist two disjoint submatrices $\mathcal{C}$ and $\mathcal{D}$ in B such that

- All entries of $\mathcal{C}$ are the same, and all entries of $\mathcal{D}$ are the same.
- All other entries of B not in $\mathcal{C}$ and $\mathcal{D}$ are 0.

If the matrices A_t are *two-block matrices* for all $t \in \{1, \ldots, m\}$ then show that $m \geq \text{rank}(A)/2$.

[3+7 = 10]

NC12. Let $\text{GL}_2(\mathbb{Z}/m\mathbb{Z})$ denote the following set of matrices

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{Z}/m\mathbb{Z} \text{ and } a_{11}a_{22} - a_{12}a_{21} \in (\mathbb{Z}/m\mathbb{Z})^* \right\}$$

with matrix multiplication as group operation. Note that $(\mathbb{Z}/m\mathbb{Z})^*$ denotes the multiplicative group of units in $\mathbb{Z}/m\mathbb{Z}$.

(i) Show that the map $\psi : \mathbb{Z}/m\mathbb{Z} \to \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$ given by $\psi(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ is an injective group homomorphism.

(ii) Show that there exists a 2×2 matrix M such that M is of order 2 in $\text{GL}_2(\mathbb{Z}/8\mathbb{Z})$ and M is of order 5 in $\text{GL}_2(\mathbb{Z}/15\mathbb{Z})$.

[3+7 = 10]

NC13. Find the perimeter of the region bounded by $x^2 + y^2 - 144$ and $x^2 + y^2 - (24\sqrt{2})x + 144 \leq 0$.