NOTATION: $\mathbb{R}$ denotes the set of all real numbers with the usual metric and topology.

1. Let M be a real $n \times n$ matrix with all diagonal entries equal to r and all non-diagonal entries equal to s. Compute the determinant of M.

2. Let $F[X]$ be the polynomial ring over a field F. Prove that the rings $F[X]/\langle X^2 \rangle$ and $F[X]/\langle X^2 - 1 \rangle$ are isomorphic if and only if the characteristic of F is 2.

3. Let C be a subset of $\mathbb{R}$ endowed with the subspace topology. If every continuous real-valued function on C is bounded, then prove that C is compact.

4. Let $A = (a_{ij})$ be a nonzero real $n \times n$ matrix such that $a_{ij} = 0$ for $i \geq j$. If $\sum_{i=0}^{k} c_i A^i = 0$ for some $c_i \in \mathbb{R}$, then prove that $c_0 = c_1 = 0$. Here A^i is the i-th power of A.

5. Let $g \colon \mathbb{R} \to \mathbb{R}$ be the function given by
$$g(x) = \begin{cases} x \sin(\frac{1}{x}), & x \neq 0; \\ 0, & x = 0. \end{cases}$$
Prove that $g(x)$ has a local maximum and a local minimum in the interval $(-\frac{1}{m}, \frac{1}{m})$ for any positive integer m.

6. Fix an integer $n \geq 1$. Suppose that n is divisible by distinct natural numbers k_1, k_2, k_3 such that
$$\gcd(k_1, k_2) = \gcd(k_2, k_3) = \gcd(k_3, k_1) = 1.$$
Pick a random natural number j uniformly from the set $\{1, 2, 3, \ldots, n\}$. Let A_d be the event that j is divisible by d. Prove that the events $A_{k_1}, A_{k_2}, A_{k_3}$ are mutually independent.

7. Let $f \colon [0,1] \to [0, \infty)$ be a function. Assume that there exists $M \geq 0$ such that $\sum_{i=1}^{k} f(x_i) \leq M$ for all $k \geq 1$ and for all $x_1, \ldots, x_k \in [0,1]$. Show that the set $\{x \mid f(x) \neq 0\}$ is countable.

8. Let G be a group having exactly three subgroups. Prove that G is cyclic of order p^2 for some prime p.