2022

Booklet Number: **00206** TEST CODE: PMA

Forenoon Time: 2 hours

- This test contains thirty (30) multiple-choice questions (MCQs).
- The questions are to be answered on a separate *Optical Mark Recognition* (OMR) Answer Sheet.
- Please write your *Name, Registration Number, Test Centre, Test Code* and the *Number of this Question Booklet* in the appropriate places on the OMR Answer Sheet. Please do not forget to put your signature in the designated place.
- For each of the questions there are four suggested answers, of which only one is correct. For each question, indicate your choice of the correct answer by darkening the appropriate circle (●) completely on the OMR Answer Sheet, using ball-point pen with BLACK ink only.
- You will score

  4 marks for each correctly answered question,

  0 mark for each incorrectly answered question, and

  1 mark for each unattempted question.
- ALL ROUGH WORK MUST BE DONE ONLY IN THE SPACE AVAILABLE ON THIS QUESTION BOOKLET.
- USE OF CALCULATORS, MOBILE PHONES AND ALL TYPES OF ELECTRONIC DEVICES IS STRICTLY PROHIBITED.

STOP! WAIT FOR THE SIGNAL TO START.

- $\mathbb{N}$ denotes the set of all positive integers.
- $\mathbb{Z}$ denotes the set of all integers.
- $\mathbb{R}$ denotes the set of all real numbers.
- $M_n(\mathbb{R})$ denotes the set of all $n \times n$ matrices with real entries.
- I_n denotes the $n \times n$ identity matrix.
- $\dim(V)$ denotes the dimension of a finite dimensional vector space V.
- For a function $f : X \to Y$, $f(X)$ denotes the set $\{f(x) : x \in X\}$.

$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$

1. If A belongs to $M_3(\mathbb{R})$, then which of the following statements is not true in general?

(A) If A is diagonalizable, then A^2 is diagonalizable.

(B) If A^2 is diagonalizable, then A is diagonalizable.

not always? — (C) There exists a polynomial p of degree 3 such that $p(A) = 0$.

× — (D) A has a real eigenvalue.

2. Let V be the vector space over $\mathbb{R}$ consisting of all polynomials with real coefficients. Consider the subset W of all polynomials

$0 \notin W$ → of degree 4. Which of the following statements is correct?

(A) W is not a subspace of V.

(B) W is a subspace of V having dimension 5.

(C) W is a subspace of V having dimension 4.

(D) W is a subspace of V having dimension 3.

0 is not degree 4 polynomial
no identity,
not subspace?

PP. = P

3. Let V be a finite dimensional vector space and $P : V \to V$ be a non-zero linear transformation such that $P^2 = P$. Which of the following statements is not true in general?

(A) $\mathrm{Ker}\,(P) \cap \mathrm{Ker}\,(I - P) = \{0\}$. true

(B) $(I - P)^2 = I - P$.

(C) $\mathrm{Ker}\,(P) = (I - P)(V)$. dim of Im P = rank

(D) $\dim\left(\mathrm{Ker}\,(P)\right) = \dim\left(P(V)\right)$. rank + nullity = dim V

4. Let W denote the vector space over $\mathbb{R}$ consisting of all polynomials of degree less than or equal to 6 with real coefficients. Let $D : W \to W$ be the linear transformation defined by

$$D(p) = \frac{dp}{dx}.$$

$D(p) = \frac{d}{dx} P$ → also a polynomial in W.

Which of the following statements is correct? $D(p_1) = \frac{d}{dx}(2x+3) = 2$

(A) D is invertible. - D is invertible

(B) Rank of D is equal to 6.

(C) Rank of D is equal to 5.

(D) Rank of D is equal to 1.

5. Let A be a 2×2 real matrix such that $(1, 0)$ is an eigenvector of A with eigenvalue $\log 2$, and $(0, 1)$ is an eigenvector of A with eigenvalue $\log 3$. Let I denote the 2×2 identity matrix. Then the limit in $\mathbb{R}^2$ of the vectors

$$\lim_{n\to\infty}\left(I + \frac{A}{n}\right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\lambda_1 = \log 2$
$\lambda_2 = \log 3.$

$\lim \left(I + \frac{A}{n}\right)^n \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(A) exists and equals $(2, 3)$.

(B) does not exist.

(C) exists and equals $(3, 5)$.

(D) exists and equals $(2, 5)$.

6. Let $W = \{A \in M_4(\mathbb{R}) : \text{Rank}\,(A) \leq 3\}$. Which of the following statements is correct?

(A) W is not closed under scalar multiplication.

(B) If B, C belong to W, then $B + C$ belongs to W.

(C) If $B \in W$, then $I_4 + B \in W$.

(D) W is not a subspace of $M_4(\mathbb{R})$.

7. Consider the following vector spaces over $\mathbb{R}$:

$$\begin{aligned} V &= \{f : [0,1] \to \mathbb{R} \mid f \text{ is a continuous function}\}, \\ W &= \{f : [0,1] \to \mathbb{R} \mid f \text{ is a function}\}. \end{aligned}$$

lossy structure

If $T : V \to W$ is the map defined by

$T(f(t)) = tf(t)$

$$(T \circ f)(t) = (Tf)(t) = tf(t), \quad t \in [0,1],$$

$T(f(t)) = (t) f(t)$

some constant on $[0,1]$

then which of the following statements is not true?

(A) $T(V) \subseteq V$.

(B) T is a linear map from V to V.

(C) T is one-to-one.

(D) $T(V) = V$.

linear map : ?

$T(a+b) = T(a) + T(b)$

$T(ca) = cT(a)$

$(T \circ f)(a+b) = t f(a+b) \stackrel{?}{=} T \circ f(a) + T \circ f(b)$?

$t f(a) + t f(b)$?

8. For positive integers r and n with $r < n$, let P denote the $n \times n$ matrix

$$\begin{bmatrix} I_r & 0_{r \times (n-r)} \\ 0_{(n-r) \times r} & 0_{(n-r) \times (n-r)} \end{bmatrix},$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

where $0_{k \times l}$ denotes the zero matrix with k rows and l columns. Let H^t denote the transpose of a matrix H and

$r + r = n$

$2r = n$

$$V = \{H \in M_n(\mathbb{R}) : H = HP + PH = H^t\}.$$

Which of the following statements is true?

(A) $\dim(V) = r^2$.

(B) $\dim(V) = (n-r)^2$.

(C) $\dim(V) = r(n-r)$.

(D) $\dim(V) = 2r(n-r)$.

9. Consider the group $G = \mathbb{Z}_6 \times \mathbb{Z}_{15} \times \mathbb{Z}_{20}$. The order of the element $(1, 2, 3)$ in G is

(A) 30. (B) 40. (C) 60. (D) 120.

10. Let $\phi : G \to H$ be a group homomorphism between finite groups. Which of the following statements is not true in general?

(A) Image of any subgroup of G under ϕ is a subgroup of H.

(B) For every $g \in G$, order of $\phi(g)$ divides the order of g.

(C) $\phi(G)$ is a subgroup of H.

(D) Image of any normal subgroup of G under ϕ is a normal subgroup of H.

11. Let p be a prime number. Consider the ring $R = \mathbb{Z}[x]/(x^2 - px)$ with multiplicative identity denoted by 1_R. Then, the number of ring homomorphisms $\phi : R \to \mathbb{Z}$ such that $\phi(1_R) = 1$ is

(A) 0. (B) 1. (C) 2. (D) 3.

?

12. Consider the ring

$\mathbb{Z}[2022i] = \{a + 2022bi : a, b \in \mathbb{Z}\}$

$1 \cdot 1 = 1$

$$\mathbb{Z}[\sqrt{-2022}] = \{a + b\sqrt{-2022} : a, b \in \mathbb{Z}\}.$$

$a = 1$, $b = 0$ $(1, 0)$ $(-1, 0)$ $(0, 0)$ X

when $b = 0$
$a \in \mathbb{Z}$.
$-a \in \mathbb{Z}$.
have additive inverse

The number of elements in this ring having multiplicative inverse is

(A) 2. (B) 3. (C) 4. (D) infinite.

$\boxed{2022^2} b^2 = \frac{1}{a^2}$

$a^2 + 2022^2 b^2 = 1$

$(a + 2022bi)(a - 2022bi) = 1$

$a^2 +- (2022bi)^2 = 1$

13. For a prime number p and $k \in \mathbb{N}$, let $\overline{k}$ denote the equivalence class of k in the field $\mathbb{Z}_p$. Consider the following system of linear equations in $\mathbb{Z}_p$:

$$\begin{aligned} \overline{10}X + \overline{5}Y &= \overline{8} \\ \overline{3}X + \overline{12}Y &= \overline{11}. \end{aligned}$$

For which of the following values of p does the above system have a unique solution?

(A) $p = 3$. (B) $p = 5$. (C) $p = 7$. (D) $p = 11$.

14. Let R be a commutative ring with unity and $a, b \in R$. Consider the ideal

inner product $I = \langle a, b \rangle$

$I + I = xa + yb + xa + yb$
$= 2xa + 2yb$
$= 2I$

$$I = \langle a, b \rangle = \{xa + yb : x \in R, y \in R\}.$$

$I^2 = \{a_1b_1 + a_2b_2 + a_nb_n\}$

$$I^2 = \left\{ \sum_{i=1}^{n} a_i b_i : a_i, b_i \in I, n \in \mathbb{N} \right\},$$

$(a-5b, b)$

then which of the following statements is not true in general?

(A) $I = \langle a - 5b, b \rangle$.
(B) $I^2 = \langle a^2, b^2 \rangle$. $I^2 = \{xa^2 + yb^2\}$
(C) $I^2 \subseteq I$.
(D) $I + I = I$.

15. Let G be a group of order 30. Which of the following statements is necessarily true?

$\mathbb{Z}_{30} \simeq \cancel{\mathbb{Z}}$ $30 = 2.3.5$

$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

(A) G is abelian but may not be cyclic.
(B) G is cyclic.
(C) G cannot have more than 7 subgroups of order 5.
(D) G can simultaneously have one subgroup of order 2, five subgroups of order 3, and six subgroups of order 5.

$= \frac{97}{36} = \sqrt{3} = \frac{16+81}{36}$ $\left(\frac{2^4+3^4}{6^2}\right)^{1/2}$ $\frac{4+9}{6}$ $= \left(\frac{13}{6}\right) = 2\ldots$

1.74

16. Let G be a group of order 75 which has an element of order 25. Then the number of elements of order 5 in G is

(A) 4. (B) 5. (C) 12. (D) 24.

17. The total number of subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is

(A) 4. (B) 12. (C) 16. (D) 22.

18. If

$$\lim_{n\to\infty} \left(\frac{2^{2n}+3^{2n}}{6^n}\right)^{1/n} = L,$$

$\lim_{n\to\infty} \left(\frac{2^n}{3^n} + \frac{3^n}{2^n}\right)^{1/n} = L$

then L is equal to

1.33 (A) $\frac{4}{3}$. (B) $\frac{13}{12}$. (C) $\frac{7}{6}$. 1.5 (D) $\frac{3}{2}$.

19. If

$$a_n = \frac{(3+(-1)^n)^n}{4^n},$$

$a_1 = \frac{1}{2}$, $a_2 = 1$, $a_3 = \left(\frac{1}{2}\right)^3$

then the power series $\sum_{n=0}^{\infty} a_n x^n$ converges if and only if

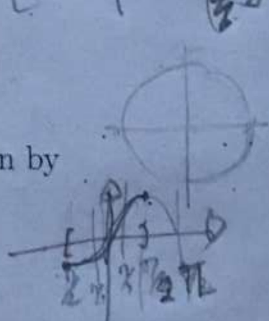
geometric series

(A) $x = 0$. yes but iff? (B) $x \in [-1, 1)$.
(C) $x \in (-1, 1]$. (D) $x \in (-1, 1)$.

20. Consider the sequence $\{f_n\}_{n=1}^{\infty}$ of functions on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ given by

$$f_n(x) = (\sin x)^n \cos x.$$

odd even = odd function



The sequence $\{f_n\}$ converges uniformly

(A) on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(B) on $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(C) on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ but not on any strictly larger subset of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(D) on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

end points?

$\frac{x_3}{x_2} = \frac{\frac{1+2^a+3^a}{3^{a+1}}}{\frac{1+2^a}{2^{a+1}}}$ $\qquad \frac{x_2}{x_1} = \frac{\frac{1+2^a}{2^{a+1}}}{1} = \frac{1+2^a}{2^{a+1}}$ $\qquad x_1 = \frac{1}{1^a} = 1 \quad x_3 = \frac{1+2^a+3^a}{3^{a+1}}$

$x_2 = \frac{1+2^a}{2^{a+1}}$

$\frac{1+2^a+3^a}{1+2^a} \cdot \frac{2^{a+1}}{3^{a+1}}$

$\frac{1+2^a+3^a}{1+2^a}\left(\frac{2}{3}\right)^{a+1}$

21. For any $a \in [0,1]$, the limit of the sequence $\{x_n\}_{n=1}^{\infty}$, where

has to go to 0 $\qquad \forall\, a \in [0,1]$

$$x_n = \frac{1+2^a+\ldots+n^a}{n^{a+1}}$$

$\frac{1}{n^{a+1}}$ when $a=1$ by p test $\frac{1}{n^2}$ converges to 0.

is

looks like a geometric series

for $a=0$, $\frac{1}{n^{a+1}} = \frac{1}{n}$ diverges

(A) $1 - \frac{a}{2}$. $\qquad$ (B) $\frac{1}{1+a^2}$.

(C) $\frac{1}{1+a}$. $\qquad$ (D) $\frac{1}{2a^2-a+1}$.

deg m $\quad$ deg n

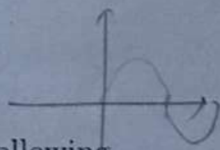
22. Let f and g be polynomials with real coefficients of degree m and n, respectively. Suppose $g(x) \neq 0$ for all $x \in [1,\infty)$. The integral

$$\int_1^{\infty} \frac{|f(x)|}{|g(x)|}\,dx$$

converges if and only if

(A) $n - m \geq 2$. $\qquad$ (B) $m - n \leq 2$.

(C) $n - m \geq 1$. $\qquad$ (D) $m \geq 1$ and $n \geq 1$.



23. Let $a_n = \frac{n}{2n+1}\sin\left(\frac{2\pi}{3}n\right)$, $n \in \mathbb{N}$. Which of the following statements is true?

$a_1 = \frac{1}{3}\sin\left(\frac{2\pi}{3}\right)$ $\qquad \sin 2\pi = 0$

(A) $\lim_{n\to\infty} a_n = \frac{1}{2}$. $\qquad a_2 = \frac{2}{5}\sin\left(\frac{4\pi}{3}\right)$

(B) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{4}$. $\qquad a_3 = \frac{3}{7}\sin(2\pi) = 0$

(C) $\limsup_{n\to\infty} a_n = \frac{\sqrt{3}}{2}$. $\qquad a_4 = \frac{4}{9}\sin\left(\frac{8\pi}{3}\right)$

(D) $\limsup_{n\to\infty} a_n = 0$. $\qquad a_5 = \frac{5}{11}\sin$

$a_6 = \frac{6}{13}\sin(4\pi) = 0$

$x_0 = 2x_1 - x_2$
$x_1 = 2x_2 - x_3$
$x_n = 2x_{n+1} - x_{n+2}$

$x_2 = 2x_1 - x_0$
$x_3 = 2x_2 - x_1$

x_0, x_1 fixed

when $a = 0$
$x_2 = x_0$
$x_3 = x_1$
$x_4 = x_2$
$x_5 = x_3 = x_1$
oscillating.

$x_2 = a x_1 + (1-a) x_0$
$x_3 = a x_2 + (1-a) x_1$
$x_4 = a x_3 + (1-a) x_2$
if $a = 1$, we get a constant function yes convergent.

24. Fix $x_0, x_1 \in \mathbb{R}$ with $x_0 \neq x_1$. Define

$x_{n+1} = 2x_n - x_{n-1}$

$$x_{n+1} = a x_n + (1-a) x_{n-1}, \quad n \in \mathbb{N}.$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is convergent if and only if

(A) $a \in (0, 2)$.
(B) $a \in [0, 2)$.
(C) $a \in [0, 2]$.
(D) $a \in (0, 1)$.

$e^{(x-1)^2}$

25. If

$$\lim_{\epsilon \to 0^+} \int_{\epsilon}^{2\epsilon} \frac{e^{(x-1)^2}}{x} dx = L,$$

then L is equal to

(A) 2.
(B) $e \log 2$.
(C) $\log(e+2)$.
(D) e^2.

26. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous and satisfies

$$|f'(x) - e^{2x}| \leq 3 \quad \text{for all } x \in \mathbb{R}.$$

$(f'(x) - e^{2x}) \leq 3$
$-3 \leq f'(x) - e^{2x} \leq 3$
$-3 + e^{2x} \leq f'(x) \leq 3 + e^{2x}$

Then the limit

$$\lim_{x \to +\infty} \frac{f(x)}{e^{2x}}$$

(A) may not exist.
(B) exists and equals $\frac{1}{2}$.
(C) exists and equals $\frac{3}{4}$.
(D) exists and equals $\frac{1}{4}$.

27. The following series of functions

$$\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}, \quad x \in \mathbb{R},$$

converges uniformly on

(A) $\mathbb{R} \setminus (a,b)$ for all $a \geq 0$.

(B) intervals of the form $(-\infty, a)$ and (a, ∞) for all $a \in \mathbb{R}$.

(C) $\mathbb{R} \setminus (a,b)$ for all $a < 0$ and $b > 0$.

(D) all compact subsets of $\mathbb{R}$.

d>0

d is symmetric

triangle inequality

compact = bounded + closed

28. Let (X, d) be a compact metric space. Let $x \in X$ be such that $X \setminus \{x\} = \{y \in X : y \neq x\}$ is complete. Which of the following statements is necessarily true?

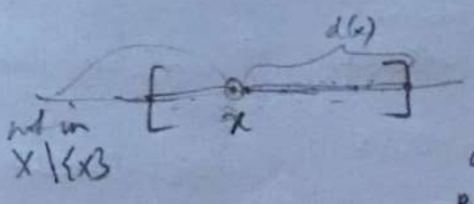
(A) There is a sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to x such that $d(x_n, x) > 0$ for all $n \in \mathbb{N}$.

(B) There exists $p \in X$ such that $0 < d(p,x) = \inf_{q \in X \setminus \{x\}} d(q,x)$.

(C) Given a point $p \in X \setminus \{x\}$, there is a point $q \neq p$ such that $d(p,x) = d(q,x)$.

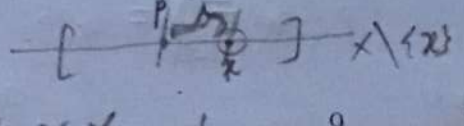
(D) For every $p \in X \setminus \{x\}$, there is a point $q \in X$ such that $0 < d(q,x) < d(p,x)$.

What about limit points?



$\exists p \in X$

$q \in X \setminus \{x\}$



$\exists p \in X$ s.t. $d(p,x) = \inf_{q \in X \setminus \{x\}} d(q,x)$

$☆ - t - t$

$☆ - 2t$

29. Let (X, d) be a metric space and $x_0 \in X$. Let $\gamma : [0, \infty) \to X$ be such that

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in [0, \infty).$$

let $\gamma(☆) = x_0$

Define a function $f : [0, \infty) \to \mathbb{R}$ by

$|0 - t| \to t \in [0, \infty]$

$$f(t) = d(x_0, \gamma(t)) - d(\gamma(0), \gamma(t)), \quad t \in [0, \infty).$$

$|☆ - t| - t$

$[0,\infty]$ $2t\,[0,\infty)$

$f(t) = ☆ - 2t$ when $\gamma(☆) = x_0$.

getting larger

Which of the following statements is necessarily true?.

(A) The function f is monotone increasing and bounded.

(B) The function f is monotone increasing and unbounded.

(C) The function f is monotone decreasing and bounded.

(D) The function f is monotone decreasing and unbounded.

30. The probability that a person tests positive for Covid by RTPCR, given that the person is infected with Covid, is 0.95. Also, the probability that a person tests positive for Covid by RTPCR, given that the person is not infected with Covid, is 0.15. Assume that in a country, 10% of the population is infected with Covid. A person is randomly chosen and the RTPCR test yields positive result. The probability that the person is actually infected with Covid is

(A) $\frac{19}{20}$. (B) $\frac{17}{46}$. (C) $\frac{19}{46}$. (D) $\frac{9}{10}$.

---

given

$P(I) = \frac{10}{100} = \frac{1}{10}$

A = Pos. = positive
B = I = infected $\bar{I}$ not infected).

$P(\text{Pos}/I) = 0.95$

$P(\text{Pos}/\bar{I}) = 0.15.$

$\cancel{P(A/B) = P(A \cap B) \cdot P(A)}$

$P(A/B) = \frac{P(B/A) \cdot P(A)}{P(B)}$

person is randomly chosen, and -ve $P(\text{pos})$.

what is $P(I)$

$P(I)$

$P(B) = \frac{P(B/A) \cdot P(A)}{P(A/B)}$

2022

Booklet Number: 00201

TEST CODE: PMB

Afternoon

Time: 2 hours

Read the following carefully before answering the test.

- **There are 8 questions in all.**
- **Each question carries 10 marks.**
- **Answer as many questions as you can.**

- Please write your *Registration Number, Test Centre, Test Code* and the *Number of this Question Booklet* in the designated places on the cover page of the Answer Booklet.
- Please use pens with black/blue ink to answer the questions.
- ALL ROUGH WORK MUST BE DONE ONLY ON THE ANSWER BOOKLET.
- USE OF CALCULATORS, MOBILE PHONES AND ALL TYPES OF ELECTRONIC DEVICES IS STRICTLY PROHIBITED.

STOP! WAIT FOR THE SIGNAL TO START.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that f' is continuous, and there exist $a, b \in \mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = a \quad \text{and} \quad \lim_{x\to\infty} f'(x) = b.$$

Show that $b = 0$.

2. Suppose V is a finite dimensional real vector space, and

$$T : V \to V$$

is a linear transformation such that $\operatorname{Rank}(T^2) = \operatorname{Rank}(T)$.

(a) Prove that $\operatorname{Ker}(T) = \operatorname{Ker}(T^2)$.

(b) Prove that $\operatorname{Ker}(T) \cap T(V) = \{0\}$.

Here $T(V) = \{T(v) : v \in V\}$.

3. Let V be a finite dimensional real vector space and suppose P_1, P_2 are two non-zero linear transformations from V to V such that the following conditions hold:

- $P_1 + P_2 = I$, I is the identity map from V to V.
- $P_1P_2 = P_2P_1 = 0$.

Prove the following statements:

(a) $P_i^2 = P_i$, for $i = 1, 2$.

(b) Let $i \in \{1, 2\}$. Then prove that the following statements are equivalent:

(i) $v \in P_i(V)$.

(ii) $P_iv = v$.

(c) $P_1(V) \cap P_2(V) = \{0\}$.

4. Let H be a normal subgroup of a group G such that there is a group homomorphism $\pi : G \to H$, with $\pi(h) = h$ for all $h \in H$. Prove that G is isomorphic to $H \times G/H$.

5. Let $\mathbb{Z}$ denote the ring of integers, and

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}.$$

   (a) Prove that any maximal ideal of $\mathbb{Z}[\sqrt{-5}]$ contains a prime number $p \in \mathbb{Z}$.

   (b) Deduce that if M is a maximal ideal of $\mathbb{Z}[\sqrt{-5}]$, then $\frac{\mathbb{Z}[\sqrt{-5}]}{M}$ is a finite field.

6. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued continuous functions on $\mathbb{R}$ such that

$$f_n\left(x + \frac{1}{n}\right) = f_n(x) \quad \text{for all } x \in \mathbb{R} \text{ and } n \geq 1.$$

Suppose there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$. Show that f is a constant function.

(You may use the following fact: If $f_n \to f$ uniformly on $\mathbb{R}$ and $x_n \to x$, then $f_n(x_n) \to f(x)$.)

7. Let $f : [0,1] \to \mathbb{R}$ be a continuous function.

(a) For $0 < a < 1$, show that

$$\lim_{n\to\infty} \int_0^a (n+1)x^n f(x)\,dx = 0.$$

(b) If $f(1) = 0$, then show that

$$\lim_{n\to\infty} \int_0^1 (n+1)x^n f(x)\,dx = 0.$$

8. Let (X, d) be a metric space. Let $B(X)$ denote the real vector space of bounded real-valued functions on X, that is,

$$B(X) = \Big\{ f : X \to \mathbb{R} \mid \text{There exists } M \text{ (depending on } f) \text{ such that } |f(x)| \le M \text{ for all } x \in X \Big\}.$$

For $f \in B(X)$, define

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

Fix an $x_0 \in X$. For each $x \in X$, let $\phi_x : X \to \mathbb{R}$ be the function

$$\phi_x(y) = d(x, y) - d(x_0, y) \quad \text{for all } y \in X.$$

(a) Show that $\phi_x \in B(X)$ for all $x \in X$.

(b) Show that $\|\phi_x - \phi_y\| = d(x, y)$ for all $x, y \in X$.