1. The sum of all the solutions of $2 + \log_2(x-2) = \log_{(x-2)} 8$ in the interval $(2, \infty)$ is

(A) $\frac{35}{8}$. (B) 5. (C) $\frac{49}{8}$. (D) $\frac{55}{8}$.

2. The value of

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots 2021}$$

is

(A) $\frac{2021}{1010}$. (B) $\frac{2021}{1011}$. (C) $\frac{2021}{1012}$. (D) $\frac{2021}{1013}$.

3. The number of ways one can express $2^2 3^3 5^5 7^7$ as a product of two numbers a and b, where $gcd(a, b) = 1$, and $1 < a < b$, is

(A) 5. (B) 6. (C) 7. (D) 8.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x+1) = \frac{1}{2} f(x) \text{ for all } x \in \mathbb{R},$$

and let $a_n = \int_0^n f(x)\, dx$ for all integers $n \geq 1$. Then:

(A) $\lim_{n\to\infty} a_n$ exists and equals $\int_0^1 f(x)\, dx$.

(B) $\lim_{n\to\infty} a_n$ does not exist.

(C) $\lim_{n\to\infty} a_n$ exists if and only if $|\int_0^1 f(x)\, dx| < 1$.

(D) $\lim_{n\to\infty} a_n$ exists and equals $2\int_0^1 f(x)\, dx$.

5. Let $a, b, c, d > 0$, be any real numbers. Then the maximum possible value of $cx + dy$, over all points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, must be

(A) $\sqrt{a^2c^2 + b^2d^2}$. (B) $\sqrt{a^2b^2 + c^2d^2}$.

(C) $\sqrt{\frac{a^2c^2+b^2d^2}{a^2+b^2}}$. (D) $\sqrt{\frac{a^2b^2+c^2d^2}{c^2+d^2}}$.

6. Let $f(x) = \sin x + \alpha x$, $x \in \mathbb{R}$, where α is a fixed real number. The function f is one-to-one if and only if

(A) $\alpha > 1$ or $\alpha < -1$. (B) $\alpha \geq 1$ or $\alpha \leq -1$.

(C) $\alpha \geq 1$ or $\alpha < -1$. (D) $\alpha > 1$ or $\alpha \leq -1$.

7. The volume of the region $S = \{(x, y, z) : |x| + 2|y| + 3|z| \leq 6\}$ is

(A) 36. (B) 48. (C) 72. (D) 6.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $\frac{d^2 f(x)}{dx^2}$ is positive for all $x \in \mathbb{R}$, and suppose $f(0) = 1, f(1) = 4$. Which of the following is not a possible value of $f(2)$?

(A) 7. (B) 8. (C) 9. (D) 10.

9. Let

$$f(x) = e^{-|x|},\ x \in \mathbb{R}\,,$$

and

$$g(\theta) = \int_{-1}^{1} f\left(\frac{x}{\theta}\right)\, dx,\ \theta \neq 0\,.$$

Then,

$$\lim_{\theta \to 0} \frac{g(\theta)}{\theta}$$

(A) equals 0. (B) equals $+\infty$.
(C) equals 2. (D) does not exist.

10. Consider the curves $x^2 + y^2 - 4x - 6y - 12 = 0,\ 9x^2 + 4y^2 - 900 = 0$ and $y^2 - 6y - 6x + 51 = 0$. The maximum number of disjoint regions into which these curves divide the XY-plane (excluding the curves themselves), is

(A) 4. (B) 5. (C) 6. (D) 7.

11. A box has 13 distinct pairs of socks. Let p_r denote the probability of having at least one matching pair among a bunch of r socks drawn at random from the box. If r_0 is the maximum possible value of r such that $p_r < 1$, then the value of p_{r_0} is

(A) $1 - \frac{12}{{}^{26}C_{12}}$. (B) $1 - \frac{13}{{}^{26}C_{13}}$. (C) $1 - \frac{2^{13}}{{}^{26}C_{13}}$. (D) $1 - \frac{2^{12}}{{}^{26}C_{12}}$.

12. Consider the following two subsets of $\mathbb{C}$:

$$A = \left\{ \tfrac{1}{z} \ : \ |z| = 2 \right\} \text{ and } B = \left\{ \tfrac{1}{z} \ : \ |z-1| = 2 \right\}.$$

Then

(A) A is a circle, but B is not a circle.

(B) B is a circle, but A is not a circle.

(C) A and B are both circles.

(D) Neither A nor B is a circle.

13. Let a, b, c and d be four non-negative real numbers where $a+b+c+d = 1$. The number of different ways one can choose these numbers such that $a^2 + b^2 + c^2 + d^2 = \max\{a, b, c, d\}$ is

(A) 1. (B) 5. (C) 11. (D) 15.

14. Suppose $f(x)$ is a twice differentiable function on $[a, b]$ such that

$$f(a) = 0 = f(b)$$

and

$$x^2 \frac{d^2 f(x)}{dx^2} + 4x \frac{df(x)}{dx} + 2f(x) > 0 \text{ for all } x \in (a, b).$$

Then,

(A) f is negative for all $x \in (a, b)$.

(B) f is positive for all $x \in (a, b)$.

(C) $f(x) = 0$ for exactly one $x \in (a, b)$.

(D) $f(x) = 0$ for at least two $x \in (a, b)$.

15. The polynomial $x^4 + 4x + c = 0$ has at least one real root if and only if

(A) $c < 2$. (B) $c \leq 2$. (C) $c < 3$. (D) $c \leq 3$.

16. The number of different ways to colour the vertices of a square $PQRS$ using one or more colours from the set {Red, Blue, Green, Yellow}, such that no two adjacent vertices have the same colour is

(A) 36. (B) 48. (C) 72. (D) 84.

17. Define $a = p^3 + p^2 + p + 11$ and $b = p^2 + 1$, where p is any prime number. Let $d = gcd\,(a, b)$. Then the set of possible values of d is

(A) $\{1, 2, 5\}$. (B) $\{2, 5, 10\}$. (C) $\{1, 5, 10\}$. (D) $\{1, 2, 10\}$.

18. Consider all 2×2 matrices whose entries are distinct and taken from the set $\{1, 2, 3, 4\}$. The sum of determinants of all such matrices is

(A) 24. (B) 10. (C) 12. (D) 0.

19. Let $f : \mathbb{R} \to \mathbb{R}$ be any twice differentiable function such that its second derivative is continuous and

$$\frac{df(x)}{dx} \neq 0 \text{ for all } x \neq 0\,.$$

If

$$\lim_{x \to 0} \frac{f(x)}{x^2} = \pi\,,$$

then

(A) for all $x \neq 0$, $f(x) > f(0)$.

(B) for all $x \neq 0$, $f(x) < f(0)$.

(C) for all x, $\frac{d^2 f(x)}{dx^2} > 0$.

(D) for all x, $\frac{d^2 f(x)}{dx^2} < 0$.

20. The number of all integer solutions of the equation $x^2 + y^2 + x - y = 2021$ is

(A) 5. (B) 7. (C) 1. (D) 0.

21. The number of different values of a for which the equation $x^3 - x + a = 0$ has two identical real roots is

(A) 0. (B) 1. (C) 2. (D) 3.

22. For a positive integer n, the equation

$$x^2 = n + y^2, \quad x, y \text{ integers,}$$

does not have a solution if and only if

(A) $n = 2$.

(B) n is a prime number.

(C) n is an odd number.

(D) n is an even number not divisible by 4.

23. For $0 \leq x < 2\pi$, the number of solutions of the equation

$$\sin^2 x + 2 \cos^2 x + 3 \sin x \cos x = 0$$

is

(A) 1. (B) 2. (C) 3. (D) 4.

24. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that

$$f(x + y) = f(x)f(y)\,,$$

for all $x, y \in \mathbb{R}$. Suppose that f is differentiable at $x = 1$ and

$$\left.\frac{df(x)}{dx}\right|_{x=1} = 2\,.$$

Then, the value of $f(1) \log_e f(1)$ is

(A) e. (B) 2. (C) $\log_e 2$. (D) 1.

25. The expression

$$\sum_{k=0}^{10} 2^k \tan(2^k)$$

equals

(A) $\cot 1 + 2^{11}\cot(2^{11})$.
(B) $\cot 1 - 2^{10}\cot(2^{10})$.
(C) $\cot 1 + 2^{10}\cot(2^{10})$.
(D) $\cot 1 - 2^{11}\cot(2^{11})$.

26. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} (1 - \cos x)\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then,

(A) f is discontinuous.

(B) f is continuous but not differentiable.

(C) f is differentiable and its derivative is discontinuous.

(D) f is differentiable and its derivative is continuous.

27. If the maximum and minimum values of $\sin^6 x + \cos^6 x$, as x takes all real values, are a and b, respectively, then $a - b$ equals

(A) $\frac{1}{2}$. (B) $\frac{2}{3}$. (C) $\frac{3}{4}$. (D) 1.

28. If two real numbers x and y satisfy $(x+5)^2 + (y-10)^2 = 196$, then the minimum possible value of $x^2 + 2x + y^2 - 4y$ is

(A) $271 - 112\sqrt{5}$.
(B) $14 - 4\sqrt{5}$.
(C) $276 - 112\sqrt{5}$.
(D) $9 - 4\sqrt{5}$.

29. Let us denote the fractional part of a real number x by $\{x\}$ (note: $\{x\} = x - [x]$ where $[x]$ is the integer part of x). Then,

$$\lim_{n\to\infty} \left\{(3 + 2\sqrt{2})^n\right\}$$

(A) equals 0.
(B) equals 1.
(C) equals $\frac{1}{2}$.
(D) does not exist.

30. Let

$$p(x) = x^3 - 3x^2 + 2x,\ x \in \mathbb{R}\,,$$

$$f_0(x) = \begin{cases} \int_0^x p(t)dt, & x \geq 0\,, \\ -\int_x^0 p(t)dt, & x < 0\,, \end{cases}$$

$$f_1(x) = e^{f_0(x)}\,, \quad f_2(x) = e^{f_1(x)}, \quad \ldots \quad , f_n(x) = e^{f_{n-1}(x)}\,.$$

How many roots does the equation $\dfrac{df_n(x)}{dx} = 0$ have in the interval $(-\infty, \infty)$?

(A) 1. (B) 3. (C) $n + 3$. (D) $3n$.