- $\bullet$  N denotes the set of positive integers.
- $\bullet \mathbb{Z}$  denotes the set of integers.
- $\bullet$   $\mathbb R$  denotes the set of real numbers.
- ullet C denotes the set of complex numbers.
- **Q1.** Find the values of a > 0 for which the improper integral

$$\int_0^\infty \frac{\sin x}{x^a} \ dx$$

converges.

**Q2.** Let  $f:[0,1] \to \mathbb{R}$  be a real-valued continuous function which is differentiable on (0,1) and satisfies f(0)=0. Suppose there exists a constant  $c \in (0,1)$  such that

$$|f'(x)| \le c|f(x)|$$
 for all  $x \in (0,1)$ .

Show that f(x) = 0 for all  $x \in [0, 1]$ .

- **Q3.** Let G be an abelian group of order n.
  - (a) If  $f: G \to \mathbb{C}$  is a function, then prove that for all  $h \in G$ ,

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(hg).$$

(b) Let  $\mathbb{C}^*$  be the multiplicative group of non-zero complex numbers and suppose  $f:G\to\mathbb{C}^*$  is a homomorphism. Prove that

$$\sum_{g \in G} f(g) = 0 \quad \text{or} \quad \sum_{g \in G} f(g) = n.$$

(c) If  $f: G \to \mathbb{C}^*$  is any homomorphism, then prove that

$$\sum_{g \in G} |f(g)| = n.$$

- **Q4.** (a) Is the ideal I = (X + Y, X Y) in the polynomial ring  $\mathbb{C}[X, Y]$  a prime ideal? Justify your answer.
  - (b) Is the ideal I = (X + Y, X Y) in the polynomial ring  $\mathbb{Z}[X, Y]$  a prime ideal? Justify your answer.

**Q5.** Let  $n \geq 2$  and A be an  $n \times n$  matrix with real entries. Let  $\operatorname{Adj} A$  denote the adjoint of A, that is, the (i, j)-th entry of  $\operatorname{Adj} A$  is the (j, i)-th cofactor of A.

Show that the rank of Adj A is 0, 1 or n.

**Q6.** Suppose an urn contains a red ball and a blue ball. A ball is drawn at random and a ball of the same colour is added to the urn along with the one that was drawn. This process is repeated indefinitely.

Let X denote the random variable that takes the value n if the first n-1 draws yield red balls and the n-th draw yields a blue ball.

- (a) If  $n \ge 1$ , find P(X > n).
- (b) Show that the probability of a blue ball being chosen eventually is 1.
- (c) Find E[X].
- **Q7.** A real number  $x_0$  is said to be a limit point of a set  $S \subseteq \mathbb{R}$  if every neighbourhood of  $x_0$  contains a point of S other than  $x_0$ . Consider the set

$$S = \{0\} \cup \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}.$$

- (a) Show that S contains infinitely many limit points of S.
- (b) Show that S is a compact subset of  $\mathbb{R}$ .
- (c) Find all limit points of S.
- **Q8.** (a) Let  $\{f_n\}_{n=1}^{\infty}$  be a sequence of continuous functions on [0,1] such that  $\sum_{n=1}^{\infty} f_n$  converges uniformly on (0,1]. Show that  $\sum_{n=1}^{\infty} f_n(0)$  converges.
  - (b) Find the set D of all points  $x \in [0,1]$  such that the series

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$$\sum_{n=1}^{\infty} e^{-nx} \cos nx$$

converges. Does this series converge uniformly on D? Justify your answer.