- $\mathbb{N}$ denotes the set of positive integers.
- $\mathbb{Z}$ denotes the set of integers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{C}$ denotes the set of complex numbers.

Q1. Find the values of $a>0$ for which the improper integral

$$
\int_{0}^{\infty} \frac{\sin x}{x^{a}} d x
$$

converges.

Q2. Let $f:[0,1] \rightarrow \mathbb{R}$ be a real-valued continuous function which is differentiable on $(0,1)$ and satisfies $f(0)=0$. Suppose there exists a constant $c \in(0,1)$ such that

$$
\left|f^{\prime}(x)\right| \leq c|f(x)| \quad \text { for all } x \in(0,1)
$$

Show that $f(x)=0$ for all $x \in[0,1]$.

Q3. Let $G$ be an abelian group of order $n$.
(a) If $f: G \rightarrow \mathbb{C}$ is a function, then prove that for all $h \in G$,

$$
\sum_{g \in G} f(g)=\sum_{g \in G} f(h g) .
$$

(b) Let $\mathbb{C}^{*}$ be the multiplicative group of non-zero complex numbers and suppose $f: G \rightarrow \mathbb{C}^{*}$ is a homomorphism. Prove that

$$
\sum_{g \in G} f(g)=0 \quad \text { or } \quad \sum_{g \in G} f(g)=n .
$$

(c) If $f: G \rightarrow \mathbb{C}^{*}$ is any homomorphism, then prove that

$$
\sum_{g \in G}|f(g)|=n .
$$

Q4. (a) Is the ideal $I=(X+Y, X-Y)$ in the polynomial ring $\mathbb{C}[X, Y]$ a prime ideal? Justify your answer.
(b) Is the ideal $I=(X+Y, X-Y)$ in the polynomial ring $\mathbb{Z}[X, Y]$ a prime ideal? Justify your answer.

Q5. Let $n \geq 2$ and $A$ be an $n \times n$ matrix with real entries. Let $\operatorname{Adj} A$ denote the adjoint of $A$, that is, the $(i, j)$-th entry of $\operatorname{Adj} A$ is the $(j, i)$-th cofactor of $A$.
Show that the rank of $\operatorname{Adj} A$ is 0,1 or $n$.

Q6. Suppose an urn contains a red ball and a blue ball. A ball is drawn at random and a ball of the same colour is added to the urn along with the one that was drawn. This process is repeated indefinitely.
Let $X$ denote the random variable that takes the value $n$ if the first $n-1$ draws yield red balls and the $n$-th draw yields a blue ball.
(a) If $n \geq 1$, find $P(X>n)$.
(b) Show that the probability of a blue ball being chosen eventually is 1 .
(c) Find $E[X]$.

Q7. A real number $x_{0}$ is said to be a limit point of a set $S \subseteq \mathbb{R}$ if every neighbourhood of $x_{0}$ contains a point of $S$ other than $x_{0}$. Consider the set

$$
S=\{0\} \cup\left\{\frac{1}{m}+\frac{1}{n}: m, n \in \mathbb{N}\right\} .
$$

(a) Show that $S$ contains infinitely many limit points of $S$.
(b) Show that $S$ is a compact subset of $\mathbb{R}$.
(c) Find all limit points of $S$.

Q8. (a) Let $\left\{f_{n}\right\}_{n=1}^{\infty}$ be a sequence of continuous functions on $[0,1]$ such that $\sum_{n=1}^{\infty} f_{n}$ converges uniformly on $(0,1]$. Show that $\sum_{n=1}^{\infty} f_{n}(0)$ converges.
(b) Find the set $D$ of all points $x \in[0,1]$ such that the series

$$
\sum_{n=1}^{\infty} e^{-n x} \cos n x
$$

converges. Does this series converge uniformly on $D$ ? Justify your answer.

