Notations: The set of integers $\{\cdots, -2, -1, 0, 1, 2\cdots\}$ is denoted by $\mathbb{Z}$. The set of real numbers is denoted by $\mathbb{R}$.

1. There are three cities each of which has exactly the same number of citizens, say n. Every citizen in each city has exactly a total of $n+1$ friends in the other two cities. Show that there exist three people, one from each city, such that they are friends. We assume that friendship is mutual (that is, a symmetric relation).

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0 = f(1)$. Assume also that f satisfies equations **(A)** and **(B)** below.

$$f(xy) = f(x) + f(y) - f(x)f(y) \qquad \textbf{(A)}$$
$$f(x-y)f(x)f(y) = f(0)f(x)f(y) \qquad \textbf{(B)}$$

for all integers x, y.
(i) Determine explicitly the set $\{f(a) : a \in \mathbb{Z}\}$.
(ii) Assuming that there is a non-zero integer a such that $f(a) \neq 0$, prove that the set $\{b : f(b) \neq 0\}$ is infinite.

3. Prove that every positive rational number can be expressed uniquely as a finite sum of the form

$$a_1 + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!},$$

where a_n are integers such that $0 \leq a_n \leq n-1$ for all $n > 1$.

4. Let $g : (0, \infty) \to (0, \infty)$ be a differentiable function whose derivative is continuous, and such that $g(g(x)) = x$ for all $x > 0$. If g is not the identity function, prove that g must be strictly decreasing.

5. Let $a_0, a_1, \cdots, a_{19} \in \mathbb{R}$ and

$$P(x) = x^{20} + \sum_{i=0}^{19} a_i x^i, \quad x \in \mathbb{R}.$$

If $P(x) = P(-x)$ for all $x \in \mathbb{R}$, and

$$P(k) = k^2, \text{ for } k = 0, 1, 2\cdots, 9,$$

then find

$$\lim_{x\to 0} \frac{P(x)}{\sin^2 x}.$$

6. If a given equilateral triangle Δ of side length a lies in the union of five equilateral triangles of side length b, show that there exist four equilateral triangles of side length b whose union contains Δ.

7. Let a, b, c be three real numbers which are roots of a cubic polynomial, and satisfy $a + b + c = 6$ and $ab + bc + ac = 9$. Suppose $a < b < c$. Show that

$$0 < a < 1 < b < 3 < c < 4.$$

8. A pond has been dug at the Indian Statistical Institute as an inverted truncated pyramid with a square base (see figure below). The depth of the pond is 6m. The square at the bottom has side length 2m and the top square has side length 8m. Water is filled in at a rate of $\frac{19}{3}$ cubic meters per hour. At what rate is the water level rising exactly 1 hour after the water started to fill the pond?