1. Let $\{x_j\}_{j\geq 1}$ be a sequence of positive numbers in geometric progression. Let P_n denote the product of the first n terms of $\{x_j\}$. Which of the following statements is true for all $n \geq 1$?

(A) $P_{3n} = P_n P_{2n}$

(B) $P_{3n} = P_{2n}^{1/2} P_n^2$

(C) $P_{3n} = P_{2n}^3 / P_n^3$

(D) $P_{3n} = P_{2n}^2 / P_n$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable non-decreasing function whose derivative f' is continuous. Fix $a < b$, and let

$$S = \{x \in (a, b) : f'(x) = 0\}.$$

Then, $f(a) < f(b)$ if and only if

(A) S is a nonempty finite set.

(B) S is a countably infinite set.

(C) S is a proper subset of (a, b).

(D) S is the empty set.

3. Let $x > 1$. Then $\lim_{n \to \infty} x^{-n} n^{(-1)^n x}$

(A) equals 0.

(B) equals e^x.

(C) equals $+\infty$.

(D) does not exist.

$x > 1$

$\lim_{n \to \infty} x^{-n} n^{(-1)^n x}$

suppose $n = 1$, $x = 2$

$\lim_{n \to 1} 2^{-1} \times 1^{(-1)^1 \times 2}$

4. Let f and g be monotonic functions on a closed interval $[a, b]$. Consider the following statements.

(I) $f + g$ is monotonic on $[a, b]$.

(II) fg is monotonic on $[a, b]$.

(III) The maxima of $f + g$ is attained at either a or b.

(IV) The maxima of fg is attained at either a or b.

Which of the above statements are correct?

(A) Only (II) and (IV).

(B) Only (I) and (III).

(C) None of (I), (II), (III), (IV).

(D) Only (III) and (IV).

5. What is the range of the function

$$f(x) = \left(x^2 + \frac{1}{2}\right) e^{-x^2}, \ x \in (-\infty, \infty)?$$

(A) $\left[\frac{1}{2}, e^{-1/2}\right]$ (B) $\left[0, e^{-1/2}\right]$ (C) $\left(0, e^{-1/2}\right]$ (D) $\left(0, \frac{1}{2}\right]$

6. Let (a, b) be a pair of real numbers. Consider the function

$$f(x) = \begin{cases} -ax - b, & \text{if } x \leq -1, \\ a^2x^2 + b, & \text{if } -1 < x \leq 1, \\ 5x^2 + 1, & \text{if } x > 1. \end{cases}$$

For how many distinct pairs (a, b) is the function f continuous everywhere?

(A) 0 (B) Exactly 2 (C) Exactly 1 (D) ∞

7. Consider a continuous function $f : [-1, 1] \to \mathbb{R}$ which is differentiable everywhere in the interval $(-1, 1)$. Further, suppose that $f(-1) = -\frac{1}{2}$ and $f'(x) \leq 1$ for all $x \in (-1, 1)$. Which of the following statements is **false**?

(A) $f(1)$ can be greater than 2.

(B) $f(1)$ can be less than -2.

(C) $f(1)$ can be greater than 1 but not greater than 2.

(D) $f(1)$ can be less than -1 but not less than -2.

8. For $\theta \in \mathbb{R}$ with $\theta \neq 0$, what is the value of

$$\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{\theta x - \theta^2 x^2/2}\, dx\,?$$

(A) $\frac{1}{\theta|\theta|}$ (B) $\frac{1}{\theta^2}$ (C) $\frac{\sqrt{e}}{\theta^2}$ (D) $\frac{\sqrt{e}}{\theta|\theta|}$

$\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{\theta x - \theta^2 x^2/2}\, dx$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underset{A}{x}\, \underset{B}{e^{\theta x(1-\theta x)/2}}\, dx$

9. Let S be the set of all real numbers a such that the matrix $M = \begin{bmatrix} 1 & -2 \\ 8 & a \end{bmatrix}$ has no real eigenvalue. Then,

(A) S is a proper subset of the interval $(-7, 9)$.

(B) $S = (-9, 7)$.

(C) The interval $(-9, 7)$ is a proper subset of S.

(D) $S = (-7, 9)$.

10. Let $\mathcal{M}$ be the set of all $n \times n$ real matrices. Which of the following statements is **false**?

(A) $AB = I$ if and only if $BA = I$ for all $A, B \in \mathcal{M}$.

(B) $\det(AB) = \det(BA)$ for all $A, B \in \mathcal{M}$.

(C) $\text{trace}(AB) = \text{trace}(BA)$ for all $A, B \in \mathcal{M}$.

(D) $\text{rank}(AB) = \text{rank}(BA)$ for all $A, B \in \mathcal{M}$.

11. Let $M(x)$ denote the matrix

$$\begin{pmatrix} x & 0 & 1 \\ 0 & 1 & 0 \\ x & 0 & -x \end{pmatrix},$$

where x is a real number. Consider the function

$$F(x) = \text{trace}\left(M(x)\, M(x)^T\right)$$

where A^T denotes the transpose of the matrix A. Which of the following statements is true?

(A) The range of F has a finite lower bound.

(B) The range of F contains 1.

(C) The range of F contains 0.

(D) The range of F has a finite upper bound.

12. Saurabh went to an amusement park with 9 tokens. There are 4 different rides available, with 3 of them costing 2 tokens each and one costing 5 tokens. Saurabh can take each ride as many times as he likes as long as he has tokens available. In how many different ways can he choose the rides so that he has no tokens left, if the order in which the rides were taken is ignored?

(A) 6 (B) 12 (C) 7 (D) 15

13. A six digit number is chosen at random. What is the probability that at least two consecutive digits in the chosen number are equal?

(A) $1 - (0.9)^5$ (B) $1 - (0.9)^6$ (C) $(0.9)^5$ (D) $(0.9)^6$

14. A multiple-choice test consists of 20 questions. Each question has four choices, exactly one of which is correct. For each question, a student is able to correctly identify one of the choices as wrong, and chooses one answer at random from the remaining three choices. The student will get a scholarship if she answers at least 18 questions correctly. What is the probability that she gets the scholarship?

(A) $1771/3^{20}$ (B) $801/4^{20}$ (C) $1771/4^{20}$ (D) $801/3^{20}$

15. The probability that a lie detector correctly determines whether a person is lying or telling the truth is 0.8, independently of whether the person is actually lying. Assume that people tell the truth 95% of the times. Given that for a particular person the lie detector says that the person is telling the truth, what is the conditional probability that he is actually telling the truth?

(A) $\frac{76}{77}$ 0.99 (B) 0.95 (C) 0.8 (D) $\frac{36}{37}$ 0.9

p(lie detector correct) = 0.8

16. Suppose that X has a uniform distribution on $\{1, 2, \ldots, n\}$. Further suppose that conditioned on the event $\{X = x\}$, Y has a uniform distribution on $\{1, 2, \ldots, x\}$. Let

$$p_y(x) = P(X = x \mid Y = y).$$

Then for each $y = 1, 2, \ldots, n$,

(A) $p_y(x) = 1/n$ for $x = 1, 2, \ldots, n$.

(B) $p_y(x) < p_y(x+1)$ for $x = y, y+1, \ldots, n-1$.

(C) $p_y(x) > p_y(x+1)$ for $x = y, y+1, \ldots, n-1$.

(D) $p_y(x) = 1/(n-y+1)$ for $x = y, y+1, \ldots, n$.

17. For $n \geq 3$, let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independently and identically distributed bivariate normal random variables with parameters $(0, 0, 1, 1, \rho)$. Suppose $(i_1, \ldots, i_n)$ is a random permutation of $1, 2, \ldots, n$ such that all $n!$ permutations are equally likely. What is the expected value of $T = \sum_{k=1}^{n} X_k Y_{i_k}$?

(A) $\frac{\rho}{1-\rho}$ (B) ρ (C) 0 (D) $\sqrt{n}\,\rho$

$E(3(3-1)(3-2)) = E(4(4-1)(4-2))$

$E(6) = E(24)$

18. Suppose X follows a Poisson distribution with probability mass function f satisfying $f(3) = f(4)$. What is $\mathrm{E}[X(X-1)(X-2)]$?

(A) 27 (B) 4 (C) 64 (D) 3

19. Let X be a random variable with the Student's t-distribution with 1 degree of freedom. Let $M(t)$ be its moment generating function. Which of the following statements is correct?

(A) $\mathrm{E}(X^{\alpha})$ is defined for all $\alpha \in (0, 1)$ but not for $\alpha = 1$.

(B) $M(t)$ is defined only on $(-a, a)$ for some $a \in (0, \infty)$.

(C) $M(t)$ is defined for all real numbers t.

(D) $\mathrm{E}(X^{\alpha})$ is defined for all $\alpha \in (0, 2)$ but not for $\alpha = 2$.

20. Which of the following distributions has a unique mode at $\frac{1}{2}$?

(A) Gamma distribution with scale parameter 1 and shape parameter $\frac{1}{2}$.

(B) Gamma distribution with scale parameter 1 and shape parameter $\frac{3}{2}$.

(C) Beta distribution on $(0, 1)$ with parameters $\frac{1}{2}$ and $\frac{1}{2}$.

(D) Beta distribution on $(0, 1)$ with parameters 1 and 1.

21. Suppose $X_1, X_2, \ldots, X_{100}$ are independent and identically distributed as Bernoulli(p), where $p \in (0,1)$. If $Y = \prod_{i=1}^{100} X_i$ and $Z = \min\{X_1, X_2, \ldots, X_{100}\}$, then

(A) Y and Z do not have the same mean but have the same variance.

(B) Y and Z have the same mean but not the same variance.

(C) Y and Z have the same distribution.

(D) Y and Z neither have the same mean nor the same variance.

of $Y = 2, \pi(2,3,4) = 24$

$Z = \min(2,3,4) = 2.$

22. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed bounded random variables with common mean 3 and common variance 5. For $n \geq 1$, define

$\mu = 3, \sigma^2 = 5$
$n \geq 1$

$$Y_n = (X_1 - X_2)^2 + (X_3 - X_4)^2 + \ldots + (X_{2n-1} - X_{2n})^2.$$

Then $\frac{1}{2n} Y_n$ converges in probability to

(A) 0.

(B) 20.

(C) 5.

(D) 14.

23. Let X be a random variable. Define the function

$$g(x) = \frac{\mathrm{P}(X < x) + \mathrm{P}(X \le x)}{2}, \quad \text{for all } x \in \mathbb{R}.$$

Which of the following statements is always true?

(A) $\lim_{x \to \infty} g(x) = 1$.

(B) g is not a monotone function.

(C) g is right continuous at all $x \in \mathbb{R}$.

(D) g is left continuous at all $x \in \mathbb{R}$.

24. Consider positive numbers $x_1, x_2, \ldots, x_n$. Let $\hat{\theta}_1$ be the value of θ that minimizes $\sum_{i=1}^{n} [\log(x_i/\theta)]^2$, and $\hat{\theta}_2$ be the value of θ that minimizes $\sum_{i=1}^{n} [(x_i - \theta)^2/x_i]$. If g and h are the geometric mean and the harmonic mean of $x_1, x_2, \ldots, x_n$, respectively, then,

(A) $\hat{\theta}_1 \neq g$ but $\hat{\theta}_2 = h$.

(B) $\hat{\theta}_1 \neq g$ and $\hat{\theta}_2 \neq h$.

(C) $\hat{\theta}_1 = g$ but $\hat{\theta}_2 \neq h$.

(D) $\hat{\theta}_1 = g$ and $\hat{\theta}_2 = h$.

$\Sigma|i^2 - \hat{\beta} i| \le \Sigma|i^2 - \beta i|$.

$0+2+6+12$

25. Let $x_i = i$ and $y_i = i^2$ for $i = 1, 2, 3, 4$. Which value of $\hat{\beta}$ satisfies

$$\sum_{i=1}^{4} |y_i - \hat{\beta} x_i| \le \sum_{i=1}^{4} |y_i - \beta x_i| \text{ for all } \beta \in \mathbb{R}?$$

(A) 5/2 (B) 3 (C) 2 (D) 10/3

26. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the uniform distribution on $(0, \theta)$, and let $Y_i = \log X_i$ for $i = 1, 2, \ldots, n$. Which of the following statistics is an unbiased estimator of $\log \theta$?

(A) $\log(2\overline{X})$

(B) $\overline{Y} + 1$

(C) $\overline{Y} - 1$

(D) $\max(Y_1, Y_2, \ldots, Y_n)$

27. Suppose $X_1, X_2, \ldots, X_5$ is a random sample from a Poisson distribution with mean λ. Consider a test for $H_0 : \lambda = 0.2$ vs $H_1 : \lambda = 0.4$ that rejects H_0 if and only if $\overline{X} > 0.4$. What is the probability of type 2 error of this test?

p(Rej Accept H_0 / H_1 is true)

$P(\overline{X} > 0.4) / \lambda = 0.4)$

(A) $3\,e^{-2}$ (B) $1.4\,e^{-0.4}$ (C) $5\,e^{-2}$ (D) $1.44\,e^{-0.4}$

28. Suppose a single observation X is obtained from a distribution with a probability density function f, which is either f_0 or f_1, where

$$f_0(x) = \begin{cases} 2x & \text{if } x \in (0,1), \\ 0 & \text{if } x \notin (0,1), \end{cases}$$

and

$$f_1(x) = \begin{cases} 5x^4 & \text{if } x \in (0,1), \\ 0 & \text{if } x \notin (0,1). \end{cases}$$

What is the maximum possible power of a level α test, based on this observation, for $H_0 : f = f_0$ against $H_1 : f = f_1$?

(A) $\sqrt{(1-\alpha)^5}$ (B) $1 - \sqrt{(1-\alpha)^5}$ (C) $\sqrt{\alpha^5}$ (D) $1 - \sqrt{\alpha^5}$

29. Suppose $Y_1, Y_2, \ldots, Y_5$ is a random sample from the exponential distribution with mean μ. Which of the following intervals contains μ with $100(1-\alpha)\%$ confidence? Here $x_{k,p}$ is the number such that $P(\chi_k^2 > x_{k,p}) = p$, and $T = \sum_{i=1}^{5} Y_i$.

(A) $\left(0, \frac{T}{x_{5,1-\alpha}}\right)$

(B) $\left(0, \frac{2x_{10,\alpha}}{T}\right)$

(C) $\left(0, \frac{2T}{x_{10,1-\alpha}}\right)$

(D) $\left(0, \frac{5T}{-\log(1-\alpha)}\right)$

30. Let X be normally distributed with mean θ and variance 1, where $\theta \in \mathbb{R}$. Assume that $\phi(X)$ is a most powerful (MP) test of size α (with $0 < \alpha < 1$) for testing $H_0 : \theta = 0$ against $H_1 : \theta = 2$. Which of the following statements is correct?

(A) For testing $H_0 : \theta = 0$ against $\widetilde{H}_1 : \theta = -2$, ϕ is a MP test of size α.

(B) For testing $\widetilde{H}_0 : \theta = -1$ against $H_1 : \theta = 2$, ϕ is a test of size α.

(C) For testing $\widetilde{H}_0 : \theta = -1$ against $H_1 : \theta = 2$, ϕ is a MP test of size α', where $\alpha' < \alpha$.

(D) For testing $\widetilde{H}_0 : \theta = 1/2$ against $H_1 : \theta = 2$, ϕ is a test of size α', where $\alpha' < \alpha$.