Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of **30 Multiple Choice Questions (MCQ)**. Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ)**. Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for this type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

NOTATION

1. $\mathbb{N} = \{1, 2, 3, \cdots\}$
2. $\mathbb{R}$ - the set of all real numbers
3. $\mathbb{R} \setminus \{0\}$ - the set of all non-zero real numbers
4. $\mathbb{C}$ - the set of all complex numbers
5. $f \circ g$ - composition of the functions f and g
6. f' and f'' - first and second derivatives of the function f, respectively
7. $f^{(n)}$ - n^{th} derivative of f
8. $\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$
9. $\oint\limits_C$ - the line integral over an oriented closed curve C
10. $\hat{i}, \hat{j}, \hat{k}$ - unit vectors along the Cartisean right handed rectangular co-ordinate system
11. $\hat{n}$ - unit outward normal vector
12. I - identity matrix of appropriate order
13. det(M) - determinant of the matrix M
14. M^{-1} - inverse of the matrix M
15. M^T - transpose of the matrix M
16. id - identity map
17. $\langle a \rangle$ - cyclic subgroup generated by an element a of a group
18. S_n - permutation group on n symbols
19. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$
20. $o(g)$ - order of the element g in a group

SECTION – A
MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q. 10 carry one mark each.

Q. 1 Let $s_n = 1 + \dfrac{(-1)^n}{n}, n \in \mathbb{N}$. Then the sequence $\{s_n\}$ is

(A) monotonically increasing and is convergent to 1

(B) monotonically decreasing and is convergent to 1

(C) neither monotonically increasing nor monotonically decreasing but is convergent to 1

(D) divergent

Q. 2 Let $f(x) = 2x^3 - 9x^2 + 7$. Which of the following is true?

(A) f is one-one in the interval $[-1, 1]$

(B) f is one-one in the interval $[2, 4]$

(C) f is NOT one-one in the interval $[-4, 0]$

(D) f is NOT one-one in the interval $[0, 4]$

Q. 3 Which of the following is FALSE?

(A) $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$

(B) $\lim\limits_{x\to 0^+} \dfrac{1}{xe^{1/x}} = 0$

(C) $\lim\limits_{x\to 0^+} \dfrac{\sin x}{1+2x} = 0$

(D) $\lim\limits_{x\to 0^+} \dfrac{\cos x}{1+2x} = 0$

Q. 4 Let $g : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. If $f(x, y) = g(y) + xg'(y)$, then

(A) $\dfrac{\partial f}{\partial x} + y\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial f}{\partial y}$

(B) $\dfrac{\partial f}{\partial y} + y\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial f}{\partial x}$

(C) $\dfrac{\partial f}{\partial x} + x\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial f}{\partial y}$

(D) $\dfrac{\partial f}{\partial y} + x\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial f}{\partial x}$

Q. 5 If the equation of the tangent plane to the surface $z = 16 - x^2 - y^2$ at the point $P(1, 3, 6)$ is $ax + by + cz + d = 0$, then the value of $|d|$ is

(A) 16 (B) 26 (C) 36 (D) 46

Q. 6 If the directional derivative of the function $z = y^2 e^{2x}$ at $(2, -1)$ along the unit vector $\vec{b} = \alpha\hat{i} + \beta\hat{j}$ is zero, then $|\alpha + \beta|$ equals

(A) $\dfrac{1}{2\sqrt{2}}$ (B) $\dfrac{1}{\sqrt{2}}$ (C) $\sqrt{2}$ (D) $2\sqrt{2}$

Q. 7 If $u = x^3$ and $v = y^2$ transform the differential equation $3x^5 dx - y(y^2 - x^3)dy = 0$ to $\dfrac{dv}{du} = \dfrac{\alpha u}{2(u - v)}$, then α is

(A) 4 (B) 2 (C) -2 (D) -4

Q. 8 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (-x, y)$. Then

(A) $T^{2k} = T$ for all $k \geq 1$

(B) $T^{2k+1} = -T$ for all $k \geq 1$

(C) the range of T^2 is a proper subspace of the range of T

(D) the range of T^2 is equal to the range of T

Q. 9 The radius of convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{n+2}{n}\right)^{n^2} x^n$$

is

(A) e^2 (B) $\dfrac{1}{\sqrt{e}}$ (C) $\dfrac{1}{e}$ (D) $\dfrac{1}{e^2}$

Q. 10 Consider the following group under matrix multiplication:

$$H = \left\{ \begin{bmatrix} 1 & p & q \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} : p, q, r \in \mathbb{R} \right\}.$$

Then the center of the group is isomorphic to

(A) $(\mathbb{R} \setminus \{0\}, \times)$
(B) $(\mathbb{R}, +)$
(C) $(\mathbb{R}^2, +)$
(D) $(\mathbb{R}, +) \times (\mathbb{R} \setminus \{0\}, \times)$

Q. 11 – Q. 30 carry two marks each.

Q. 11 Let $\{a_n\}$ be a sequence of positive real numbers. Suppose that $l = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Which of the following is true?

(A) If $l = 1$, then $\lim_{n\to\infty} a_n = 1$
(B) If $l = 1$, then $\lim_{n\to\infty} a_n = 0$
(C) If $l < 1$, then $\lim_{n\to\infty} a_n = 1$
(D) If $l < 1$, then $\lim_{n\to\infty} a_n = 0$

Q. 12 Define $s_1 = \alpha > 0$ and $s_{n+1} = \sqrt{\frac{1+s_n^2}{1+\alpha}}$, $n \geq 1$. Which of the following is true?

(A) If $s_n^2 < \frac{1}{\alpha}$, then $\{s_n\}$ is monotonically increasing and $\lim_{n\to\infty} s_n = \frac{1}{\sqrt{\alpha}}$

(B) If $s_n^2 < \frac{1}{\alpha}$, then $\{s_n\}$ is monotonically decreasing and $\lim_{n\to\infty} s_n = \frac{1}{\alpha}$

(C) If $s_n^2 > \frac{1}{\alpha}$, then $\{s_n\}$ is monotonically increasing and $\lim_{n\to\infty} s_n = \frac{1}{\sqrt{\alpha}}$

(D) If $s_n^2 > \frac{1}{\alpha}$, then $\{s_n\}$ is monotonically decreasing and $\lim_{n\to\infty} s_n = \frac{1}{\alpha}$

Q. 13 Suppose that S is the sum of a convergent series $\sum_{n=1}^{\infty} a_n$. Define $t_n = a_n + a_{n+1} + a_{n+2}$. Then the series $\sum_{n=1}^{\infty} t_n$

(A) diverges

(B) converges to $3S - a_1 - a_2$

(C) converges to $3S - a_1 - 2a_2$

(D) converges to $3S - 2a_1 - a_2$

Q. 14 Let $a \in \mathbb{R}$. If $f(x) = \begin{cases} (x+a)^2, & x \le 0 \\ (x+a)^3, & x > 0, \end{cases}$

then

(A) $\dfrac{d^2 f}{dx^2}$ does not exist at $x = 0$ for any value of a

(B) $\dfrac{d^2 f}{dx^2}$ exists at $x = 0$ for exactly one value of a

(C) $\dfrac{d^2 f}{dx^2}$ exists at $x = 0$ for exactly two values of a

(D) $\dfrac{d^2 f}{dx^2}$ exists at $x = 0$ for infinitely many values of a

Q. 15 Let $f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y}, & xy \neq 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0, y = 0 \\ y^2 \sin\frac{1}{y}, & y \neq 0, x = 0 \\ 0, & x = y = 0. \end{cases}$

Which of the following is true at $(0,0)$?

(A) f is not continuous

(B) $\dfrac{\partial f}{\partial x}$ is continuous but $\dfrac{\partial f}{\partial y}$ is not continuous

(C) f is not differentiable

(D) f is differentiable but both $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are not continuous

Q. 16 Let S be the surface of the portion of the sphere with centre at the origin and radius 4, above the xy-plane. Let $\vec{F} = y\hat{i} - x\hat{j} + yx^3\hat{k}$. If $\hat{n}$ is the unit outward normal to S, then

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

equals

(A) -32π (B) -16π (C) 16π (D) 32π

Q. 17 Let $f(x, y, z) = x^3 + y^3 + z^3 - 3xyz$. A point at which the gradient of the function f is equal to zero is

(A) $(-1, 1, -1)$ (B) $(-1, -1, -1)$ (C) $(-1, 1, 1)$ (D) $(1, -1, 1)$

Q. 18 The area bounded by the curves $x^2 + y^2 = 2x$ and $x^2 + y^2 = 4x$, and the straight lines $y = x$ and $y = 0$ is

(A) $3\left(\frac{\pi}{2} + \frac{1}{4}\right)$ (B) $3\left(\frac{\pi}{4} + \frac{1}{2}\right)$ (C) $2\left(\frac{\pi}{4} + \frac{1}{3}\right)$ (D) $2\left(\frac{\pi}{3} + \frac{1}{4}\right)$

Q. 19 Let M be a real 6×6 matrix. Let 2 and -1 be two eigenvalues of M. If $M^5 = aI + bM$, where $a, b \in \mathbb{R}$, then

(A) $a = 10, b = 11$ (B) $a = -11, b = 10$

(C) $a = -10, b = 11$ (D) $a = 10, b = -11$

Q. 20 Let M be an $n \times n$ $(n \geq 2)$ non-zero real matrix with $M^2 = 0$ and let $\alpha \in \mathbb{R} \setminus \{0\}$. Then

(A) α is the only eigenvalue of $(M + \alpha I)$ and $(M - \alpha I)$

(B) α is the only eigenvalue of $(M + \alpha I)$ and $(\alpha I - M)$

(C) $-\alpha$ is the only eigenvalue of $(M + \alpha I)$ and $(M - \alpha I)$

(D) $-\alpha$ is the only eigenvalue of $(M + \alpha I)$ and $(\alpha I - M)$

Q. 21 Consider the differential equation $L[y] = (y - y^2)dx + xdy = 0$. The function $f(x, y)$ is said to be an integrating factor of the equation if $f(x, y)L[y] = 0$ becomes exact.

If $f(x, y) = \dfrac{1}{x^2y^2}$, then

(A) f is an integrating factor and $y = 1 - kxy,\ k \in \mathbb{R}$ is NOT its general solution

(B) f is an integrating factor and $y = -1 + kxy,\ k \in \mathbb{R}$ is its general solution

(C) f is an integrating factor and $y = -1 + kxy,\ k \in \mathbb{R}$ is NOT its general solution

(D) f is NOT an integrating factor and $y = 1 + kxy,\ k \in \mathbb{R}$ is its general solution

Q. 22 A solution of the differential equation $2x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} - y = 0, x > 0$ that passes through the point $(1, 1)$ is

(A) $y = \dfrac{1}{x}$ (B) $y = \dfrac{1}{x^2}$ (C) $y = \dfrac{1}{\sqrt{x}}$ (D) $y = \dfrac{1}{x^{3/2}}$

Q. 23 Let M be a 4×3 real matrix and let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Which of the following is true?

(A) If rank$(M) = 1$, then $\{Me_1, Me_2\}$ is a linearly independent set

(B) If rank$(M) = 2$, then $\{Me_1, Me_2\}$ is a linearly independent set

(C) If rank$(M) = 2$, then $\{Me_1, Me_3\}$ is a linearly independent set

(D) If rank$(M) = 3$, then $\{Me_1, Me_3\}$ is a linearly independent set

Q. 24 The value of the triple integral $\iiint\limits_V (x^2y+1)\, dxdydz$, where V is the region given by $x^2+y^2 \leq 1, 0 \leq z \leq 2$ is

(A) π (B) 2π (C) 3π (D) 4π

Q. 25 Let S be the part of the cone $z^2 = x^2 + y^2$ between the planes $z = 0$ and $z = 1$. Then the value of the surface integral $\iint\limits_S (x^2 + y^2)\, dS$ is

(A) π (B) $\dfrac{\pi}{\sqrt{2}}$ (C) $\dfrac{\pi}{\sqrt{3}}$ (D) $\dfrac{\pi}{2}$

Q. 26 Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, x, y, z \in \mathbb{R}$. Which of the following is FALSE?

(A) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$

(B) $\nabla \cdot (\vec{a} \times \vec{r}) = 0$

(C) $\nabla \times (\vec{a} \times \vec{r}) = \vec{a}$

(D) $\nabla \cdot ((\vec{a} \cdot \vec{r})\vec{r}) = 4(\vec{a} \cdot \vec{r})$

Q. 27 Let $D = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \le 1\}$ and $f : D \to \mathbb{R}$ be a non-constant continuous function. Which of the following is TRUE?

(A) The range of f is unbounded

(B) The range of f is a union of open intervals

(C) The range of f is a closed interval

(D) The range of f is a union of at least two disjoint closed intervals

Q. 28 Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f\left(\frac{1}{2}\right) = -\frac{1}{2}$ and

$$|f(x) - f(y) - (x - y)| \le \sin\left(|x - y|^2\right)$$

for all $x, y \in [0, 1]$. Then $\int_0^1 f(x)\, dx$ is

(A) $-\frac{1}{2}$

(B) $-\frac{1}{4}$

(C) $\frac{1}{4}$

(D) $\frac{1}{2}$

Q. 29 Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group under multiplication and $i = \sqrt{-1}$. Then the set $\{\theta \in \mathbb{R} : \langle e^{i2\pi\theta} \rangle$ is infinite$\}$ is

(A) empty

(B) non-empty and finite

(C) countably infinite

(D) uncountable

Q. 30 Let $F = \{\omega \in \mathbb{C} : \omega^{2020} = 1\}$. Consider the groups

$$G = \left\{ \begin{pmatrix} \omega & z \\ 0 & 1 \end{pmatrix} : \omega \in F, z \in \mathbb{C} \right\}$$

and

$$H = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$$

under matrix multiplication. Then the number of cosets of H in G is

(A) 1010 (B) 2019 (C) 2020 (D) infinite

SECTION – B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q. 31 Let $a, b, c \in \mathbb{R}$ such that $a < b < c$. Which of the following is/are true for any continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(a) = b$, $f(b) = c$ and $f(c) = a$?

(A) There exists $\alpha \in (a, c)$ such that $f(\alpha) = \alpha$

(B) There exists $\beta \in (a, b)$ such that $f(\beta) = \beta$

(C) There exists $\gamma \in (a, b)$ such that $(f \circ f)(\gamma) = \gamma$

(D) There exists $\delta \in (a, c)$ such that $(f \circ f \circ f)(\delta) = \delta$

Q. 32 If $s_n = \dfrac{(-1)^n}{2^n + 3}$ and $t_n = \dfrac{(-1)^n}{4n - 1}$, $n = 0, 1, 2, \ldots$, then

(A) $\sum\limits_{n=0}^{\infty} s_n$ is absolutely convergent

(B) $\sum\limits_{n=0}^{\infty} t_n$ is absolutely convergent

(C) $\sum\limits_{n=0}^{\infty} s_n$ is conditionally convergent

(D) $\sum\limits_{n=0}^{\infty} t_n$ is conditionally convergent

Q. 33 Let $a, b \in \mathbb{R}$ and $a < b$. Which of the following statement(s) is/are true?

(A) There exists a continuous function $f : [a, b] \to (a, b)$ such that f is one-one

(B) There exists a continuous function $f : [a, b] \to (a, b)$ such that f is onto

(C) There exists a continuous function $f : (a, b) \to [a, b]$ such that f is one-one

(D) There exists a continuous function $f : (a, b) \to [a, b]$ such that f is onto

Q. 34 Let V be a non-zero vector space over a field F. Let $S \subset V$ be a non-empty set. Consider the following properties of S:

(I) For any vector space W over F, any map $f : S \to W$ extends to a linear map from V to W.

(II) For any vector space W over F and any two linear maps $f, g : V \to W$ satisfying $f(s) = g(s)$ for all $s \in S$, we have $f(v) = g(v)$ for all $v \in V$.

(III) S is linearly independent.

(IV) The span of S is V.

Which of the following statement(s) is /are true?

(A) (I) implies (IV) (B) (I) implies (III)

(C) (II) implies (III) (D) (II) implies (IV)

Q. 35 Let $L[y] = x^2 \dfrac{d^2y}{dx^2} + px\dfrac{dy}{dx} + qy$, where p, q are real constants. Let $y_1(x)$ and $y_2(x)$ be two solutions of $L[y] = 0, x > 0$, that satisfy $y_1(x_0) = 1, y_1'(x_0) = 0, y_2(x_0) = 0$ and $y_2'(x_0) = 1$ for some $x_0 > 0$. Then,

(A) $y_1(x)$ is not a constant multiple of $y_2(x)$

(B) $y_1(x)$ is a constant multiple of $y_2(x)$

(C) $1, \ln x$ are solutions of $L[y] = 0$ when $p = 1, q = 0$

(D) $x, \ln x$ are solutions of $L[y] = 0$ when $p + q \neq 0$

Q. 36 Consider the following system of linear equations

$$x + y + 5z = 3, \quad x + 2y + mz = 5 \quad \text{and} \quad x + 2y + 4z = k.$$

The system is consistent if

(A) $m \neq 4$ (B) $k \neq 5$ (C) $m = 4$ (D) $k = 5$

Q. 37 Let $a = \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{(n-1)}{n^2}\right)$ and $b = \lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right)$.
Which of the following is/are true?

(A) $a > b$ (B) $a < b$ (C) $ab = \ln\sqrt{2}$ (D) $\frac{a}{b} = \ln\sqrt{2}$

Q. 38 Let S be that part of the surface of the paraboloid $z = 16 - x^2 - y^2$ which is above the plane $z = 0$ and D be its projection on the xy-plane. Then the area of S equals

(A) $\iint\limits_D \sqrt{1 + 4(x^2 + y^2)}\, dxdy$ (B) $\iint\limits_D \sqrt{1 + 2(x^2 + y^2)}\, dxdy$

(C) $\int\limits_0^{2\pi}\int\limits_0^4 \sqrt{1 + 4r^2}\, drd\theta$ (D) $\int\limits_0^{2\pi}\int\limits_0^4 \sqrt{1 + 4r^2}\, rdrd\theta$

Q. 39 Let f be a real valued function of a real variable, such that $|f^{(n)}(0)| \leq K$ for all $n \in \mathbb{N}$, where $K > 0$. Which of the following is/are true?

(A) $\left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}} \to 0$ as $n \to \infty$

(B) $\left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}} \to \infty$ as $n \to \infty$

(C) $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$

(D) The series $\sum\limits_{n=1}^{\infty} \frac{f^{(n)}(0)}{(n-1)!}$ is absolutely convergent

Q. 40 Let G be a group with identity e. Let H be an abelian non-trivial proper subgroup of G with the property that $H \cap gHg^{-1} = \{e\}$ for all $g \notin H$.

If $K = \big\{g \in G : gh = hg \text{ for all } h \in H\big\}$, then

(A) K is a proper subgroup of H

(B) H is a proper subgroup of K

(C) $K = H$

(D) there exists no abelian subgroup $L \subseteq G$ such that K is a proper subgroup of L

SECTION – C
NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q. 41 Let $x_n = n^{\frac{1}{n}}$ and $y_n = e^{1-x_n}, n \in \mathbb{N}$. Then the value of $\lim_{n\to\infty} y_n$ is ________.

Q. 42 Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and S be the sphere given by $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$. If $\hat{n}$ is the unit outward normal to S, then

$$\frac{1}{\pi}\iint_S \vec{F}\cdot\hat{n}\, dS$$

is ________.

Q. 43 Let $f : \mathbb{R} \to \mathbb{R}$ be such that f, f', f'' are continuous functions with $f > 0, f' > 0$ and $f'' > 0$. Then

$$\lim_{x\to-\infty} \frac{f(x) + f'(x)}{2}$$

is ________.

Q. 44 Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and $f : S \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$. Then

$$\max\left\{\delta : \left|x - \frac{1}{3}\right| < \delta \implies \left|f(x) - f\left(\frac{1}{3}\right)\right| < 1\right\}$$

is ________. (rounded off to two decimal places)

Q. 45 Let $f(x, y) = e^x \sin y, x = t^3 + 1$ and $y = t^4 + t$. Then $\frac{df}{dt}$ at $t = 0$ is ________. (rounded off to two decimal places)

Q. 46 Consider the differential equation

$$\frac{dy}{dx} + 10y = f(x),\ x > 0,$$

where $f(x)$ is a continuous function such that $\lim_{x\to\infty} f(x) = 1$. Then the value of

$$\lim_{x\to\infty} y(x)$$

is _______.

Q. 47 If $\int_0^1 \int_{2y}^2 e^{x^2} dx dy = k(e^4 - 1)$, then k equals _______.

Q. 48 Let $f(x, y) = 0$ be a solution of the homogeneous differential equation

$$(2x + 5y)dx - (x + 3y)dy = 0.$$

If $f(x + \alpha, y - 3) = 0$ is a solution of the differential equation

$$(2x + 5y - 1)dx + (2 - x - 3y)dy = 0,$$

then the value of α is _______.

Q. 49 Consider the real vector space $P_{2020} = \{\sum_{i=0}^{n} a_i x^i : a_i \in \mathbb{R} \text{ and } 0 \leq n \leq 2020\}$. Let W be the subspace given by

$$W = \left\{ \sum_{i=0}^{n} a_i x^i \in P_{2020} : a_i = 0 \text{ for all odd } i \right\}.$$

Then, the dimension of W is _______.

Q. 50 Let $\phi : S_3 \to S^1$ be a non-trivial non-injective group homomorphism. Then, the number of elements in the kernel of ϕ is _______.

Q. 51 – Q. 60 carry two marks each.

Q. 51 The sum of the series $\dfrac{1}{2(2^2-1)} + \dfrac{1}{3(3^2-1)} + \dfrac{1}{4(4^2-1)} + \cdots$ is ________.

Q. 52 Consider the expansion of the function $f(x) = \dfrac{3}{(1-x)(1+2x)}$ in powers of x, that is valid in $|x| < \dfrac{1}{2}$. Then the coefficient of x^4 is ________.

Q. 53 The minimum value of the function $f(x, y) = x^2 + xy + y^2 - 3x - 6y + 11$ is ________.

Q. 54 Let $f(x) = \sqrt{x} + \alpha x,\ x > 0$ and

$$g(x) = a_0 + a_1(x-1) + a_2(x-1)^2$$

be the sum of the first three terms of the Taylor series of $f(x)$ around $x = 1$. If $g(3) = 3$, then α is ________.

Q. 55 Let C be the boundary of the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ oriented in the counter clockwise sense. Then, the value of the line integral

$$\oint_C x^2y^2dx + (x^2 - y^2)dy$$

is ________. (rounded off to two decimal places)

Q. 56 Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f'(x) = f(x)$ for all x. Suppose that $f(\alpha x)$ and $f(\beta x)$ are two non-zero solutions of the differential equation

$$4\frac{d^2y}{dx^2} - p\frac{dy}{dx} + 3y = 0$$

satisfying

$$f(\alpha x)f(\beta x) = f(2x) \ \text{ and } \ f(\alpha x)f(-\beta x) = f(x).$$

Then, the value of p is ________.

Q. 57 If $x^2 + xy^2 = c$, where $c \in \mathbb{R}$, is the general solution of the exact differential equation

$$M(x, y)\, dx + 2xy\, dy = 0,$$

then $M(1, 1)$ is _______.

Q. 58 Let $M = \begin{bmatrix} 9 & 2 & 7 & 1 \\ 0 & 7 & 2 & 1 \\ 0 & 0 & 11 & 6 \\ 0 & 0 & -5 & 0 \end{bmatrix}$. Then, the value of $\det\Big((8I - M)^3\Big)$ is _______.

Q. 59 Let $T : \mathbb{R}^7 \to \mathbb{R}^7$ be a linear transformation with $\text{Nullity}(T) = 2$. Then, the minimum possible value for $\text{Rank}(T^2)$ is _______.

Q. 60 Suppose that G is a group of order 57 which is NOT cyclic. If G contains a unique subgroup H of order 19, then for any $g \notin H$, $o(g)$ is _______.

END OF THE QUESTION PAPER