UGB
2019

Notation.

$\mathbb{R}$ denotes the set of all real numbers.
$\mathbb{C}$ denotes the set of all complex numbers.

1. Prove that the positive integers n that cannot be written as a sum of r consecutive positive integers, with $r > 1$, are of the form $n = 2^l$ for some $l \geq 0$.

2. Let $f : (0, \infty) \to \mathbb{R}$ be defined by
$$f(x) = \lim_{n\to\infty} \cos^n\left(\frac{1}{n^x}\right).$$
   (a) Show that f has exactly one point of discontinuity.
   (b) Evaluate f at its point of discontinuity.

3. Let $\Omega = \{z = x + iy \in \mathbb{C} : |y| \leq 1\}$. If $f(z) = z^2 + 2$, then draw a sketch of
$$f(\Omega) = \{f(z) : z \in \Omega\}.$$
   Justify your answer.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that
$$\frac{1}{2y}\int_{x-y}^{x+y} f(t)\,dt = f(x), \quad \text{for all } x \in \mathbb{R}, y > 0.$$
   Show that there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

5. A subset S of the plane is called *convex* if given any two points x and y in S, the line segment joining x and y is contained in S. A quadrilateral is called convex if the region enclosed by the edges of the quadrilateral is a convex set.

   Show that given a convex quadrilateral Q of area 1, there is a rectangle R of area 2 such that Q can be drawn inside R.

6. For all natural numbers n, let

$$A_n = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}} \quad (n \text{ many radicals}).$$

(a) Show that for $n \geq 2$,

$$A_n = 2 \sin \frac{\pi}{2^{n+1}}.$$

(b) Hence, or otherwise, evaluate the limit

$$\lim_{n \to \infty} 2^n A_n.$$

7. Let f be a polynomial with integer coefficients. Define

$$a_1 = f(0), a_2 = f(a_1) = f\big(f(0)\big),$$

and

$$a_n = f(a_{n-1}) \quad \text{for } n \geq 3.$$

If there exists a natural number $k \geq 3$ such that $a_k = 0$, then prove that either $a_1 = 0$ or $a_2 = 0$.

8. Consider the following subsets of the plane:

$$C_1 = \{(x, y) : x > 0, y = \tfrac{1}{x}\}$$

and

$$C_2 = \{(x, y) : x < 0, y = -1 + \tfrac{1}{x}\}.$$

Given any two points $P = (x, y)$ and $Q = (u, v)$ of the plane, their distance $d(P, Q)$ is defined by

$$d(P, Q) = \sqrt{(x - u)^2 + (y - v)^2}.$$

Show that there exists a unique choice of points $P_0 \in C_1$ and $Q_0 \in C_2$ such that

$$d(P_0, Q_0) \leq d(P, Q) \quad \text{for all } P \in C_1 \text{ and } Q \in C_2.$$