$\mathbb{N} = \{1, 2, \ldots\}$.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

$\mathbb{Q}$ = the set of rational numbers.

$\mathbb{R}$ = the set of real numbers.

$\mathbb{R}^n$ = the n-dimensional real space with the Euclidean topology.

$\mathbb{C}$ = the set of complex numbers.

$\mathbb{C}^n$ = the n-dimensional complex space with the Euclidean topology.

$M_n(\mathbb{R}), M_n(\mathbb{C})$ = the vector space of $n \times n$ real or complex matrices, respectively.

$f', \ f''$ = the first and second derivatives of the function f, respectively.

$f^{(n)}$ = the nth. derivative of the function f.

$\int_C$ stands for the line integral over the curve C.

I_n = the $n \times n$ identity matrix.

A^{-1} = the inverse of an invertible matrix A.

S_n = the permutation group on n symbols.

$\hat{i} = (1, 0, 0), \hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$.

$\ln x$ = the natural logarithm of x (to the base e).

$|X|$ = the number of elements in a finite set X.

$\mathbb{Z}_n$ = the additive group of integers modulo n.

$\arctan(x)$ denotes the unique $\theta \in (-\pi/2, \pi/2)$ such that $\tan\theta = x$.

All vector spaces are over the real or complex field, unless otherwise stated.

SECTION – A
MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q. 10 carry one mark each.

Q. 1 Let $0 < \alpha < 1$ be a real number. The number of differentiable functions $y : [0,1] \to [0,\infty)$, having continuous derivative on $[0,1]$ and satisfying

$$\begin{aligned} y'(t) &= (y(t))^{\alpha}, \quad t \in [0,1], \\ y(0) &= 0, \end{aligned}$$

is

(A) exactly one.

(B) exactly two.

(C) finite but more than two.

(D) infinite.

Q. 2 Let $P : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $P(x) > 0$ for all $x \in \mathbb{R}$. Let y be a twice differentiable function on $\mathbb{R}$ satisfying $y''(x) + P(x)y'(x) - y(x) = 0$ for all $x \in \mathbb{R}$. Suppose that there exist two real numbers a, b $(a < b)$ such that $y(a) = y(b) = 0$. Then

(A) $y(x) = 0$ for all $x \in [a, b]$.

(B) $y(x) > 0$ for all $x \in (a, b)$.

(C) $y(x) < 0$ for all $x \in (a, b)$.

(D) $y(x)$ changes sign on (a, b).

Q. 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(x) = f(x + 1)$ for all $x \in \mathbb{R}$. Then

(A) f is not necessarily bounded above.

(B) there exists a unique $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

(C) there is no $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

(D) there exist infinitely many $x_0 \in \mathbb{R}$ such that $f(x_0 + \pi) = f(x_0)$.

Q. 4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$,

$$\int_0^1 f(xt)\,dt = 0. \qquad (*)$$

Then

(A) f must be identically 0 on the whole of $\mathbb{R}$.

(B) there is an f satisfying $(*)$ that is identically 0 on $(0,1)$ but not identically 0 on the whole of $\mathbb{R}$.

(C) there is an f satisfying $(*)$ that takes both positive and negative values.

(D) there is an f satisfying $(*)$ that is 0 at infinitely many points, but is not identically zero.

Q. 5 Let p and t be positive real numbers. Let D_t be the closed disc of radius t centered at $(0,0)$, i.e., $D_t = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq t^2\}$. Define

$$I(p,t) = \iint_{D_t} \frac{dxdy}{(p^2 + x^2 + y^2)^p}.$$

Then $\lim_{t\to\infty} I(p,t)$ is finite

(A) only if $p > 1$.

(B) only if $p = 1$.

(C) only if $p < 1$.

(D) for no value of p.

Q. 6 How many elements of the group $\mathbb{Z}_{50}$ have order 10?

(A) 10 (B) 4 (C) 5 (D) 8

Q. 7 For every $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be a function. From the given choices, pick the statement that is the negation of

"For every $x \in \mathbb{R}$ and for every real number $\epsilon > 0$, there exists an integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| < \epsilon$ for every integer $p > 0$."

(A) For every $x \in \mathbb{R}$ and for every real number $\epsilon > 0$, there does not exist any integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| < \epsilon$ for every integer $p > 0$.

(B) For every $x \in \mathbb{R}$ and for every real number $\epsilon > 0$, there exists an integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| \geq \epsilon$ for some integer $p > 0$.

(C) There exists $x \in \mathbb{R}$ and there exists a real number $\epsilon > 0$ such that for every integer $N > 0$, there exists an integer $p > 0$ for which the inequality $\sum_{i=1}^{p} |f_{N+i}(x)| \geq \epsilon$ holds.

(D) There exists $x \in \mathbb{R}$ and there exists a real number $\epsilon > 0$ such that for every integer $N > 0$ and for every integer $p > 0$ the inequality $\sum_{i=1}^{p} |f_{N+i}(x)| \geq \epsilon$ holds.

Q. 8 Which one of the following subsets of $\mathbb{R}$ has a non-empty interior?

(A) The set of all irrational numbers in $\mathbb{R}$.

(B) The set $\{a \in \mathbb{R} : \sin(a) = 1\}$.

(C) The set $\{b \in \mathbb{R} : x^2 + bx + 1 = 0 \text{ has distinct roots}\}$.

(D) The set of all rational numbers in $\mathbb{R}$.

Q. 9 For an integer $k \geq 0$, let P_k denote the vector space of all real polynomials in one variable of degree less than or equal to k. Define a linear transformation $T : P_2 \longrightarrow P_3$ by

$$Tf(x) = f''(x) + xf(x).$$

Which one of the following polynomials is not in the range of T?

(A) $x + x^2$ (B) $x^2 + x^3 + 2$ (C) $x + x^3 + 2$ (D) $x + 1$

Q. 10 Let $n > 1$ be an integer. Consider the following two statements for an arbitrary $n \times n$ matrix A with complex entries.

I. If $A^k = I_n$ for some integer $k \geq 1$, then all the eigenvalues of A are k^{th} roots of unity.

II. If, for some integer $k \geq 1$, all the eigenvalues of A are k^{th} roots of unity, then $A^k = I_n$.

Then

(A) both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) I is FALSE but II is TRUE.

(D) neither I nor II is TRUE.

Q. 11 – Q. 30 carry two marks each.

Q. 11 Let $M_n(\mathbb{R})$ be the real vector space of all $n \times n$ matrices with real entries, $n \geq 2$. Let $A \in M_n(\mathbb{R})$. Consider the subspace W of $M_n(\mathbb{R})$ spanned by $\{I_n, A, A^2, \ldots\}$. Then the dimension of W over $\mathbb{R}$ is necessarily

(A) ∞. (B) n^2. (C) n. (D) at most n.

Q. 12 Let y be the solution of

$$(1+x)y''(x) + y'(x) - \frac{1}{1+x}y(x) = 0, \quad x \in (-1, \infty),$$
$$y(0) = 1, \; y'(0) = 0.$$

Then

(A) y is bounded on $(0, \infty)$. (B) y is bounded on $(-1, 0]$.

(C) $y(x) \geq 2$ on $(-1, \infty)$. (D) y attains its minimum at $x = 0$.

Q. 13 Consider the surface $S = \{(x, y, xy) \in \mathbb{R}^3 : x^2 + y^2 \leq 1\}$. Let $\vec{F} = y\hat{i} + x\hat{j} + \hat{k}$. If $\hat{n}$ is the continuous unit normal field to the surface S with positive z-component, then

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

equals

(A) $\frac{\pi}{4}$. (B) $\frac{\pi}{2}$. (C) π. (D) 2π.

Q. 14 Consider the following statements.

I. The group $(\mathbb{Q}, +)$ has no proper subgroup of finite index.

II. The group $(\mathbb{C} \setminus \{0\}, \cdot)$ has no proper subgroup of finite index.

Which one of the following statements is true?

(A) Both I and II are TRUE. (B) I is TRUE but II is FALSE.

(C) II is TRUE but I is FALSE. (D) Neither I nor II is TRUE.

Q. 15 Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective map such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^2} < +\infty.$$

The number of such bijective maps is

(A) exactly one.

(B) zero.

(C) finite but more than one.

(D) infinite.

Q. 16 Define

$$S = \lim_{n\to\infty}\Big(1 - \frac{1}{2^2}\Big)\Big(1 - \frac{1}{3^2}\Big)\cdots\Big(1 - \frac{1}{n^2}\Big).$$

Then

(A) $S = 1/2$.

(B) $S = 1/4$.

(C) $S = 1$.

(D) $S = 3/4$.

Q. 17 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function such that for all $a, b \in \mathbb{R}$ with $a < b$,

$$\frac{f(b) - f(a)}{b - a} = f'\Big(\frac{a+b}{2}\Big).$$

Then

(A) f must be a polynomial of degree less than or equal to 2.

(B) f must be a polynomial of degree greater than 2.

(C) f is not a polynomial.

(D) f must be a linear polynomial.

Q. 18 Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}, \\ 1 - \frac{1}{p} & \text{if } x = \frac{n}{p},\ n \in \mathbb{Z} \setminus \{0\},\ p \in \mathbb{N} \text{ and } \gcd(n,p) = 1. \end{cases}$$

Then

(A) all $x \in \mathbb{Q} \setminus \{0\}$ are strict local minima for f.

(B) f is continuous at all $x \in \mathbb{Q}$.

(C) f is not continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$.

(D) f is not continuous at $x = 0$.

Q. 19 Consider the family of curves $x^2 - y^2 = ky$ with parameter $k \in \mathbb{R}$. The equation of the orthogonal trajectory to this family passing through $(1, 1)$ is given by

(A) $x^3 + 3xy^2 = 4$.

(B) $x^2 + 2xy = 3$.

(C) $y^2 + 2x^2y = 3$.

(D) $x^3 + 2xy^2 = 3$.

Q. 20 Which one of the following statements is true?

(A) Exactly half of the elements in any even order subgroup of S_5 must be even permutations.

(B) Any abelian subgroup of S_5 is trivial.

(C) There exists a cyclic subgroup of S_5 of order 6.

(D) There exists a normal subgroup of S_5 of index 7.

Q. 21 Let $f : [0, 1] \to [0, \infty)$ be a continuous function such that

$$\big(f(t)\big)^2 < 1 + 2\int_0^t f(s)\, ds, \text{ for all } t \in [0, 1].$$

Then

(A) $f(t) < 1 + t$ for all $t \in [0, 1]$.

(B) $f(t) > 1 + t$ for all $t \in [0, 1]$.

(C) $f(t) = 1 + t$ for all $t \in [0, 1]$.

(D) $f(t) < 1 + \frac{t}{2}$ for all $t \in [0, 1]$.

Q. 22 Let A be an $n \times n$ invertible matrix and C be an $n \times n$ nilpotent matrix. If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is a $2n \times 2n$ matrix (each X_{ij} being $n \times n$) that commutes with the $2n \times 2n$ matrix $B = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$, then

(A) X_{11} and X_{22} are necessarily zero matrices.

(B) X_{12} and X_{21} are necessarily zero matrices.

(C) X_{11} and X_{21} are necessarily zero matrices.

(D) X_{12} and X_{22} are necessarily zero matrices.

Q. 23 Let $D \subseteq \mathbb{R}^2$ be defined by $D = \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}\}$. Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x, y) = x \sin \frac{1}{y}.$$

Then

(A) f is a discontinuous function on D.

(B) f is a continuous function on D and cannot be extended continuously to any point outside D.

(C) f is a continuous function on D and can be extended continuously to $D \cup \{(0, 0)\}$.

(D) f is a continuous function on D and can be extended continuously to the whole of $\mathbb{R}^2$.

Q. 24 Which one of the following statements is true?

(A) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{R}, +)$.

(B) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$.

(C) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}/2\mathbb{Z}, +)$.

(D) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$.

Q. 25 Let y be a twice differentiable function on $\mathbb{R}$ satisfying

$$y''(x) = 2 + e^{-|x|}, \quad x \in \mathbb{R},$$
$$y(0) = -1, \quad y'(0) = 0.$$

Then

(A) $y = 0$ has exactly one root.

(B) $y = 0$ has exactly two roots.

(C) $y = 0$ has more than two roots.

(D) there exists an $x_0 \in \mathbb{R}$ such that $y(x_0) \geq y(x)$ for all $x \in \mathbb{R}$.

Q. 26 Let $f : [0,1] \to [0,1]$ be a non-constant continuous function such that $f \circ f = f$. Define

$$E_f = \{x \in [0,1] : f(x) = x\}.$$

Then

(A) E_f is neither open nor closed.

(B) E_f is an interval.

(C) E_f is empty.

(D) E_f need not be an interval.

Q. 27 Let g be an element of S_7 such that g commutes with the element $(2, 6, 4, 3)$. The number of such g is

(A) 6. (B) 4. (C) 24. (D) 48.

Q. 28 Let G be a finite abelian group of odd order. Consider the following two statements:

I. The map $f : G \to G$ defined by $f(g) = g^2$ is a group isomorphism.

II. The product $\prod_{g \in G} g = e$.

(A) Both I and II are TRUE.

(B) I is TRUE but II is FALSE.

(C) II is TRUE but I is FALSE.

(D) Neither I nor II is TRUE.

Q. 29 Let $n \geq 2$ be an integer. Let $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the linear transformation defined by

$$A(z_1, z_2, \ldots, z_n) = (z_n, z_1, z_2, \ldots, z_{n-1}).$$

Which one of the following statements is true for every $n \geq 2$?

(A) A is nilpotent.

(B) All eigenvalues of A are of modulus 1.

(C) Every eigenvalue of A is either 0 or 1.

(D) A is singular.

Q. 30 Consider the two series

$$\text{I.} \quad \sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}} \quad \text{and} \quad \text{II.} \quad \sum_{n=1}^{\infty} \frac{1}{n^{2-n^{1/n}}}.$$

Which one of the following holds?

(A) Both I and II converge.

(B) Both I and II diverge.

(C) I converges and II diverges.

(D) I diverges and II converges.

SECTION – B
MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q. 31 Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the property that for every $y \in \mathbb{R}$, the value of the expression

$$\sup_{x \in \mathbb{R}} [xy - f(x)]$$

is finite. Define $g(y) = \sup_{x \in \mathbb{R}} [xy - f(x)]$ for $y \in \mathbb{R}$. Then

(A) g is even if f is even.

(B) f must satisfy $\lim_{|x| \to \infty} \dfrac{f(x)}{|x|} = +\infty$.

(C) g is odd if f is even.

(D) f must satisfy $\lim_{|x| \to \infty} \dfrac{f(x)}{|x|} = -\infty$.

Q. 32 Consider the equation

$$x^{2021} + x^{2020} + \cdots + x - 1 = 0.$$

Then

(A) all real roots are positive.

(B) exactly one real root is positive.

(C) exactly one real root is negative.

(D) no real root is positive.

Q. 33 Let $D = \mathbb{R}^2 \setminus \{(0, 0)\}$. Consider the two functions $u, v : D \to \mathbb{R}$ defined by

$$u(x, y) = x^2 - y^2 \text{ and } v(x, y) = xy.$$

Consider the gradients ∇u and ∇v of the functions u and v, respectively. Then

(A) ∇u and ∇v are parallel at each point (x, y) of D.

(B) ∇u and ∇v are perpendicular at each point (x, y) of D.

(C) ∇u and ∇v do not exist at some points (x, y) of D.

(D) ∇u and ∇v at each point (x, y) of D span $\mathbb{R}^2$.

Q. 34 Consider the two functions $f(x, y) = x + y$ and $g(x, y) = xy - 16$ defined on $\mathbb{R}^2$. Then

(A) the function f has no global extreme value subject to the condition $g = 0$.

(B) the function f attains global extreme values at $(4, 4)$ and $(-4, -4)$ subject to the condition $g = 0$.

(C) the function g has no global extreme value subject to the condition $f = 0$.

(D) the function g has a global extreme value at $(0, 0)$ subject to the condition $f = 0$.

Q. 35 Let $f : (a, b) \to \mathbb{R}$ be a differentiable function on (a, b). Which of the following statements is/are true?

(A) $f' > 0$ in (a, b) implies that f is increasing in (a, b).

(B) f is increasing in (a, b) implies that $f' > 0$ in (a, b).

(C) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then there exists a $\delta > 0$ such that $f(x) > f(x_0)$ for all $x \in (x_0, x_0 + \delta)$.

(D) If $f'(x_0) > 0$ for some $x_0 \in (a, b)$, then f is increasing in a neighbourhood of x_0.

Q. 36 Let G be a finite group of order 28. Assume that G contains a subgroup of order 7. Which of the following statements is/are true?

(A) G contains a unique subgroup of order 7.

(B) G contains a normal subgroup of order 7.

(C) G contains no normal subgroup of order 7.

(D) G contains at least two subgroups of order 7.

Q. 37 Which of the following subsets of $\mathbb{R}$ is/are connected?

(A) The set $\{x \in \mathbb{R} : x \text{ is irrational}\}$.

(B) The set $\{x \in \mathbb{R} : x^3 - 1 \geq 0\}$.

(C) The set $\{x \in \mathbb{R} : x^3 + x + 1 \geq 0\}$.

(D) The set $\{x \in \mathbb{R} : x^3 - 2x + 1 \geq 0\}$.

Q. 38 Consider the four functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f_1(x) = x^4 + 3x^3 + 7x + 1, \quad f_2(x) = x^3 + 3x^2 + 4x, \quad f_3(x) = \arctan(x)$$

and

$$f_4(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Which of the following subsets of $\mathbb{R}$ are open?

(A) The range of f_1.

(B) The range of f_2.

(C) The range of f_3.

(D) The range of f_4.

Q. 39 Let V be a finite dimensional vector space and $T : V \to V$ be a linear transformation. Let $\mathcal{R}(T)$ denote the range of T and $\mathcal{N}(T)$ denote the null space $\{v \in V : Tv = 0\}$ of T. If $\text{rank}(T) = \text{rank}(T^2)$, then which of the following is/are necessarily true?

(A) $\mathcal{N}(T) = \mathcal{N}(T^2)$.

(B) $\mathcal{R}(T) = \mathcal{R}(T^2)$.

(C) $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\}$.

(D) $\mathcal{N}(T) = \{0\}$.

Q. 40 Let $m > 1$ and $n > 1$ be integers. Let A be an $m \times n$ matrix such that for some $m \times 1$ matrix b_1, the equation $Ax = b_1$ has infinitely many solutions. Let b_2 denote an $m \times 1$ matrix different from b_1. Then $Ax = b_2$ has

(A) infinitely many solutions for some b_2.

(B) a unique solution for some b_2.

(C) no solution for some b_2.

(D) finitely many solutions for some b_2.

SECTION – C
NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q. 41 The number of cycles of length 4 in S_6 is _____.

Q. 42 The value of

$$\lim_{n\to\infty}\left(3^n+5^n+7^n\right)^{\frac{1}{n}}$$

is _____.

Q. 43 Let $B=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2\le 1\}$ and define $u(x,y,z)=\sin\left((1-x^2-y^2-z^2)^2\right)$ for $(x,y,z)\in B$. Then the value of

$$\iiint_B\left(\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}\right)dxdydz$$

is _____.

Q. 44 Consider the subset $S=\{(x,y): x^2+y^2>0\}$ of $\mathbb{R}^2$. Let

$$P(x,y)=\frac{y}{x^2+y^2} \text{ and } Q(x,y)=-\frac{x}{x^2+y^2}$$

for $(x,y)\in S$. If C denotes the unit circle traversed in the counter-clockwise direction, then the value of

$$\frac{1}{\pi}\int_C(Pdx+Qdy)$$

is _____.

Q. 45 Consider the set $A=\{a\in\mathbb{R} : x^2=a(a+1)(a+2) \text{ has a real root }\}$. The number of connected components of A is _____.

Q. 46 Let V be the real vector space of all continuous functions $f:[0,2]\to\mathbb{R}$ such that the restriction of f to the interval $[0,1]$ is a polynomial of degree less than or equal to 2, the restriction of f to the interval $[1,2]$ is a polynomial of degree less than or equal to 3 and $f(0)=0$. Then the dimension of V is equal to _____.

Q. 47 The number of group homomorphisms from the group $\mathbb{Z}_4$ to the group S_3 is _____.

Q. 48 Let $y : \left(\frac{9}{10}, 3\right) \to \mathbb{R}$ be a differentiable function satisfying

$$(x-2y)\frac{dy}{dx} + (2x+y) = 0, \quad x \in \left(\frac{9}{10}, 3\right), \quad \text{and } y(1) = 1.$$

Then $y(2)$ equals _____.

Q. 49 Let $\vec{F} = (y+1)e^y \cos(x)\hat{i} + (y+2)e^y \sin(x)\hat{j}$ be a vector field in $\mathbb{R}^2$ and C be a continuously differentiable path with the starting point $(0, 1)$ and the end point $(\frac{\pi}{2}, 0)$. Then

$$\int_C \vec{F} \cdot d\vec{r}$$

equals _____.

Q. 50 The value of

$$\frac{\pi}{2} \lim_{n\to\infty} \cos\left(\frac{\pi}{4}\right) \cos\left(\frac{\pi}{8}\right) \cdots \cos\left(\frac{\pi}{2^{n+1}}\right)$$

is _____.

Q. 51 – Q. 60 carry two marks each.

Q. 51 The number of elements of order two in the group S_4 is equal to _____.

Q. 52 The least possible value of k, accurate up to two decimal places, for which the following problem

$$y''(t) + 2y'(t) + ky(t) = 0, t \in \mathbb{R},$$
$$y(0) = 0, y(1) = 0, y(1/2) = 1,$$

has a solution is _____.

Q. 53 Consider those continuous functions $f : \mathbb{R} \to \mathbb{R}$ that have the property that given any $x \in \mathbb{R}$,

$$f(x) \in \mathbb{Q} \text{ if and only if } f(x+1) \in \mathbb{R} \setminus \mathbb{Q}.$$

The number of such functions is _____.

Q. 54 The largest positive number a such that

$$\int_0^5 f(x)dx + \int_0^3 f^{-1}(x)dx \geq a$$

for every strictly increasing surjective continuous function $f : [0, \infty) \to [0, \infty)$ is _____.

Q. 55 Define the sequence

$$s_n = \begin{cases} \dfrac{1}{2^n} \displaystyle\sum_{j=0}^{n-2} 2^{2j} & \text{if } n > 0 \text{ is even,} \\ \\ \dfrac{1}{2^n} \displaystyle\sum_{j=0}^{n-1} 2^{2j} & \text{if } n > 0 \text{ is odd.} \end{cases}$$

Define $\sigma_m = \dfrac{1}{m} \displaystyle\sum_{n=1}^{m} s_n$. The number of limit points of the sequence $\{\sigma_m\}$ is _____.

Q. 56 The determinant of the matrix

$$\begin{pmatrix} 2021 & 2020 & 2020 & 2020 \\ 2021 & 2021 & 2020 & 2020 \\ 2021 & 2021 & 2021 & 2020 \\ 2021 & 2021 & 2021 & 2021 \end{pmatrix}$$

is _____.

Q. 57 The value of

$$\lim_{n\to\infty} \int_0^1 e^{x^2} \sin(nx)\, dx$$

is _____.

Q. 58 Let S be the surface defined by

$$\{(x, y, z) \in \mathbb{R}^3 : z = 1 - x^2 - y^2,\ z \geq 0\}.$$

Let $\vec{F} = -y\hat{i} + (x-1)\hat{j} + z^2\hat{k}$ and $\hat{n}$ be the continuous unit normal field to the surface S with positive z-component. Then the value of

$$\frac{1}{\pi} \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS$$

is _____.

Q. 59 Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}$. Then the largest eigenvalue of A is _____.

Q. 60 Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Consider the linear map T_A from the real vector space $M_4(\mathbb{R})$ to itself defined by $T_A(X) = AX - XA,$ for all $X \in M_4(\mathbb{R})$. The dimension of the range of T_A is _____.

END OF THE QUESTION PAPER